On the Similarity of Operator Algebras to C*-Algebras

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2006

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This is an expository thesis which addresses the requirements for an operator algebra to be similar to a $\mathcal{C}^*$-algebra. It has been conjectured that this similarity condition is equivalent to either amenability or total reductivity; however, the problem has only been solved for specific types of operators.

We define amenability and total reductivity, as well as present some of the implications of these properties. For the purpose of establishing the desired result in specific cases, we describe the properties of two well-known types of operators, namely the compact operators and quasitriangular operators. Finally, we show that if $\mathfrak{A}$ is an algebra of compact operators or of triangular operators then $\mathfrak{A}$ is similar to a $\mathcal{C}^*$ algebra if and only if it has the total reduction property.

Acknowledgements

I owe a debt of gratitude to my supervisors, Dr. Brian Forrest and Dr. Laurent Marcoux, for their direction and encouragement. Many of the mistakes in the original copy were found and corrected by my readers, Dr. Heydar Radjavi and Dr. Nicolaas Spronk. Shonn, the Graduate Studies Administrative Coordinator, provided invaluable assistence in moments of stress. Likewise, for their friendship, counsel and support, I must acknowledge Emily, Stephanie, Luke, Philip and Daniel, as well as my sister Cristina, and my parents, Cristian and Nadia Georgescu. I wish to thank them all for their contributions.

Dedication

To Ion Bazavan and Bujor Georgescu

Contents

Chapter 1

Introduction

The focus of this paper is some of the known conditions under which an operator algebra is similar to a $\mathcal{C}^*$-algebra. Since the current mathematical understanding of the topic is incomplete, we discuss some of the properties of operator algebras which are involved in the partial answers to this problem, and the interplay between these properties.

One reason this topic is of interest is the connection with Kadison's similarity problem. Some time in the 50's (see [12]), Kadison posed the following question: If we have a $\mathcal{C}^*$-algebra $\mathfrak{A}$ and a representation $\phi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$, when is ϕ similar to a *-homomorphism? Clearly, if there exists a similarity S such that $a \to S\phi(a)S^{-1}$ is a *-homomorphism, then it follows that $\phi(\mathfrak{A})$ is similar to a $\mathcal{C}^*$-algebra. More importantly, the same concepts seem to be involved in the answer to both questions.

The key properties of operator algebras with which we will concern ourselves are amenability, total reductivity and complete reductivity. These concepts are related, in that amenable algebras are total reduction algebras, and in turn total reduction algebras are complete reduction algebras. The inclusion at each step is known to be strict; however, if specific additional constraints are imposed, some of these algebra classes can coincide. It is conjectured that an operator algebra $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra whenever $\mathfrak{A}$ has the total reduction property.

This is an expository thesis; its aim is to give an overview of the relevant areas of mathematics, present some of the known results, and direct the reader towards further resources. The majority of the results presented in this paper are from [20], [8] and [13]. Runde's book [20] offers a thorough introduction to the theory of amenable groups and algebras. In his thesis [8], Gifford introduces complete and total reduction algebras, building on earlier concepts of algebras with a reduction property. The interplay between the two properties is presented in detail; the thesis is also a good source of examples of algebras that have the various properties. Finally, [13] presents the results we describe for triangular algebras, as well as some generalizations which are not included in this thesis.

The concept of amenability has its roots in group theory, and as such we will introduce it from that angle in Chapter 2. We also discuss a related similarity question for groups, namely under what circumstances is a bounded representation similar to a unitary representation. The connection between amenability for groups and amenability for Banach algebras comes from Johnson's theorem, which states that a locally compact group G is amenable if and only if the algebra $L^1(G)$ is amenable.

In Chapter 3 we relate these general results to operator algebras. Since we are going to be working with specific types of operators, we list some of the properties we will need to establish our results. We also introduce the concepts of complete and total reduction algebras.

Finally, in Chapter 4, we outline the current results about similarity to $\mathcal{C}^*$-algebras. The results which are available so far deal with special cases of operator algebras. The strongest result, due to Gifford, is that if we have a subalgebra of compact operators, then it is similar to a $\mathcal{C}^*$-algebra if and only if it has the complete reduction property. The other specific result available deals with unital subalgebras of triangular operators. In this case, we have that the total reduction property, amenability and similarity to a $\mathcal{C}^*$-algebra are all equivalent.

Chapter 2

Amenable Groups and Algebras

2.1 Locally compact Groups

A brief history of amenability and its applications to various areas of mathematics is given in [14]. The concept was introduced more than a hundred years ago; while he was working on the properties of the integral which now bears his name, Lebesgue inquired into the existence of a positive, finitely additive and translation invariant measure on $\mathbb{R}$ such that the measure of the unit interval is 1. It was later noted that a finitely additive, translation invariant probability measure which is absolutely continuous with respect to the Lebesgue measure can be extended to a linear functional on $L^\infty(\mathbb{R})$. It is from the point of view of linear functionals that amenability has been defined and studied in recent years, a change in perception that allows the full power of functional analysis to bear on the concept.

This chapter introduces the subject of amenability for groups and for Banach algebras, and explains why the same terminology is used in both cases, even though the connection is not obvious from the definitions given. Most of the proofs shown follow the ones presented in "Amenable Banach Algebras" by Volker Runde [20].

We let G be a locally compact group with Haar measure μ (a proof of the fact that any locally compact group G has a left Haar measure can be found in [7], Section 2.2). For $g \in G$ and $\phi \in L^\infty(G)$ we define $\delta_g * \phi$ to be left translation by g, so $(\delta_g * \phi)(h) = \phi(g^{-1}h)$ for any $h \in G$.

Definition 2.1. *A **mean** on $L^\infty(G)$ is a linear functional $m \in L^\infty(G)^*$ such that*

$\|m\| = m(\mathbf{1}) = 1$. *We say that* m *is* ***left invariant*** *if for all* $g \in G$ *and* $\phi \in L^\infty(G)$ *we have that* $m(\delta_g * \phi) = m(\phi)$.

If we consider $L^\infty(G)$ with the multiplication operation given by pointwise multiplication of functions and involution given by $f^*(g) = \overline{f(g)}$ for $f \in L^\infty(G)$, $g \in G$, then $L^\infty(G)$ is a commutative $\mathcal{C}^*$-algebra. The above definition of mean corresponds to the definition of a state in a $\mathcal{C}^*$-algebra context. Therefore, the properties of states apply.

Consider E a subspace of $L^\infty(G)$. In order to extend the above definitions to E, we need E to contain the constant functions. For left invariant means, we also require that $\delta_g * \phi \in E$ for all $g \in G$ and $\phi \in E$ (in which case we say that E is left invariant). Finally, in order to identify the set of means with the positive functionals which evaluate to 1 at $\mathbf{1}$ (see [20], Proposition 1.1.2) we need E to be closed under complex conjugation. Two subspaces of $L^\infty(G)$ which satisfy these properties, and hence for which the definition of left invariant mean makes sense, are $C_b(G)$ (the set of continuous bounded functions on G), and $UC(G)$ (the set of uniformly continuous functions on G).

In general, if $E \subseteq L^\infty(G)$ is a subspace which has the properties described above then the set of means is non-empty. As an example let us consider $E = C_b(G)$. We can define for each $g \in G$ the function $m_g : E \to \mathbb{C}$ given by $m_g(\phi) = \phi(g)$ for each $\phi \in E$. Then m_g is linear, $m_g(\mathbf{1}) = 1$ and $|m_g(\phi)| \leq \|\phi\|_\infty$, so m_g is a mean on E. A result which will prove useful later is that the set of convex combinations of means of type m_g, $A_m = \{\sum_{i=1}^{n} k_i m_{g_i} : n \in \mathbb{N}, g_i \in G, \text{ and } k_i \geq 0 \text{ such that } \sum_{i=1}^{n} k_i = 1\}$, is weak*-dense in the set of all means on $C_b(G)$. This can be obtained as a consequence of the Hahn-Banach Separation Theorem.

Moreover, in general, the set of means is weak*-compact in E^*. This follows easily from the Banach-Alaoglu theorem, since the means are contained in the closed unit ball of E^* and the limit of a weak* convergent net of means is another mean. However, the existence of a left invariant mean is not guaranteed. We have in fact the following definition.

Definition 2.2. *G is* ***amenable*** *if there is a left invariant mean on* $L^\infty(G)$.

$\mathbb{F}_2$, the free group on two generators, is a classic example of a group which is not amenable. We will prove this result by contradiction. Suppose that m is a left invariant mean on $\mathbb{F}_2$, and denote the generators of $\mathbb{F}_2$ by a and b. Consider the set

$S = \{w \in \mathbb{F}_2 : w \text{ starts with a } b \text{ or a } b^{-1}\}$. Suppose $m(\chi_S) = \alpha$. Note that the sets S, aS and a^2S are disjoint. Hence $\chi_S + \chi_{aS} + \chi_{a^2S} \leq \mathbf{1}$; since m is a positive functional this implies that $m(\chi_S + \chi_{aS} + \chi_{a^2S}) \leq m(\mathbf{1}) = 1$. But $m(\chi_{aS}) = m(\delta_a * \chi_S) = m(\chi_S)$ since m is left invariant, and similarly $m(\chi_{a^2S}) = \alpha$. So by linearity of m we get $3\alpha \leq 1$, i.e. $\alpha \leq 1/3$. On the other hand, $\mathbb{F}_2 = S \cup bS$, so $\mathbf{1} \leq \chi_S + \chi_{bS}$. Similarly to above we obtain $1 \leq 2\alpha$, and hence $\alpha \geq 1/2$. So we have $1/2 \leq \alpha \leq 1/3$, a contradiction. Therefore, $\mathbb{F}_2$ is not amenable. However, we can show that all compact groups and all locally compact, abelian groups are amenable.

Proposition 2.3. *If G is a compact group then G is amenable.*

Proof. We assume that G is equipped with a left-invariant Haar measure. For G compact we have $L^\infty(G) \subset L^1(G)$, so we can define $m \in L^\infty(G)^*$ by $m(\phi) = \int_G \phi(g)\, dg$. It is then easy to see that m is a mean; moreover, since the Haar measure h is left-invariant, so is m. Therefore, G is amenable. $\square$

The result for locally compact, abelian groups is slightly harder to obtain. We will use the fixed point theorem given below:

Theorem 2.4 (Markov-Kakutani)**.** *Let E be a locally convex Hausdorff space, and $K \subset E$ be a compact, convex set. If $(T_\alpha)_{\alpha \in I}$ is a family of continuous, affine endomorphisms on K such that $T_\alpha T_\beta = T_\beta T_\alpha$ for all $\alpha, \beta \in I$, then all the T_α's have a common fixed point.*

Proposition 2.5. *Let G be a locally compact, abelian group. Then G is amenable.*

Proof. We know that the set of means is weak* compact in $L^\infty(G)^*$; so we can let K be the set of means. For each $g \in G$ define $T_g : K \to K$ by $[T_g(m)](\phi) = m(\delta_g * \phi)$ for $\phi \in L^\infty(G)$. It is clear that T_g is a continuous, affine endomorphism, and its range is indeed contained in K. Hence in order to use the Markov-Kakutani theorem, we just have to establish that $(T_g)_{g \in G}$ is commutative. Note that $[T_g T_h(m)](\phi) = m(\delta_h * \delta_g * \phi) = [T_{gh}(m)](\phi)$, so $T_g T_h = T_{gh}$. Since G is abelian we have $T_{gh} = T_{hg}$, i.e. $T_g T_h = T_h T_g$.

Therefore the Markov-Kakutani Fixed Point Theorem (Theorem 2.4) applies; so there is an m_0 such that $T_g(m_0) = m_0$ for all $g \in G$. But by definition $[T_g(m_0)](\phi) = m_0(\phi)$, which implies $m_0(\delta_g * \phi) = m_0(\phi)$ for all $g \in G$; in other words, m_0 is left invariant. Thus G admits a left-invariant mean, namely m_0, which proves that G is amenable. $\square$

In order to prove that a group is amenable we can exhibit an explicit left invariant mean or infer its existence from properties of the group, as in the examples above. Also, the properties of amenability can be used to draw conclusions about a group by relating it to known amenable groups. In this latter category, some examples of groups which are amenable are: closed subgroups of amenable groups, quotients of amenable groups by closed, normal subgroups and groups which have a closed normal subgroup such that both the subgroup and the quotient by that subgroup are amenable ([20], Section 1.2).

The definition of left invariant mean can be applied to a subspace of $L^\infty(G)$ as explained at the beginning of the chapter. In particular, $UC(G)$ contains uniformly continuous functions, which are easier to work with than general functions in $L^\infty(G)$. We will in fact show that if there is a left invariant mean on $UC(G)$, then G is amenable.

Recall that if $f \in L^1(G)$ and $\phi \in L^\infty(G)$ we define their convolution by $(f * \phi)(h) = \int_G f(g)\phi(g^{-1}h)\,dg$. The interaction of $L^1(G)$ functions with $L^\infty(G)$ functions will allow us to define a useful subset of the left invariant means, as shown below.

Definition 2.6. *Let $P(G) := \{f \in L^1(G) : f \geq 0$ and $\|f\|_1 = 1\}$. Let $E = L^\infty(G), C_b(G)$ or $UC(G)$. If $m \in E^*$ then m is* ***topologically left invariant*** *if $m(f * \phi) = m(\phi)$ for all $f \in P(G)$ and $\phi \in E$.*

Note in particular that for $f \in P(G)$ we can define $m_f : L^\infty(G) \to \mathbb{C}$ by $m_f(\phi) = \int_G f(g)\phi(g)\,dg$. Then m_f is a bounded linear functional that satisfies $m_f(\mathbf{1}) = 1$ and $|m_f(\phi)| \leq \|\phi\|_\infty$, so m_f is a mean on $L^\infty(G)$. Hence it follows that $P(G)$ consists exactly of those functions in $L^1(G)$ which are means (where we identify $L^1(G)$ with a subset of its second dual $L^\infty(G)^*$). Since the Haar measure has the property that the measure of the group is finite if and only if the group is compact, it is easy to check that $P(G)$ contains a left invariant mean if and only if G is compact.

We will also need the following result about bounded approximate identities on $L^1(G)$:

Theorem 2.7. *([20], Theorem A.1.8) If $\mathcal{U}$ is a basis of neighbourhoods of the identity and $\{f_\alpha\}_{\alpha\in\mathcal{U}}$ is a net in $P(G)$ such that the support of f_α is contained in α then $\{f_\alpha\}_{\alpha\in\mathcal{U}}$ is a bounded approximate identity for $L^1(G)$.*

The next theorem relates $UC(G)$ to $L^1(G)$, and allows us to use the bounded approximate identity on $L^1(G)$ for $UC(G)$, as we will explain below.

Theorem 2.8. *([20], Theorem A.2.5)* $UC(G) = \{f_1 * \phi * f_2 : f_i \in L^1(G), \phi \in L^\infty(G)\}$.

Suppose $(f_\alpha)_\alpha$ is a bounded approximate identity for $L^1(G)$ as described in Theorem 2.7. Hence, by Theorem 2.8 above, for any $\varphi \in UC(G)$ we can write $\varphi = f_1 * \phi * f_2$ for some $f_1, f_2 \in L^1(G)$ and $\phi \in L^\infty(G)$; since $f_\alpha * f_1 \to f_1$ (by the definition of a bounded approximate identity), it follows from the definition of convolution that $f_\alpha * \varphi \to \varphi$ in $UC(G)$.

Theorem 2.9. *Let G be a locally compact group, and $m \in UC(G)^*$. Then m is left invariant if and only if it is topologically left invariant.*

Proof. Suppose $m \in UC(G)^*$ is topologically left invariant. Pick any $f \in P(G)$. Then for any $g \in G$ and $\phi \in UC(G)^*$ we have

$$\begin{aligned} m(\delta_g * \phi) &= m(f * \delta_g * \phi) && \text{(since } m \text{ is topologically left invariant)} \\ &= m(\phi) && \text{(since } f * \delta_g \in P(G)) \end{aligned}$$

Thus m is left invariant.

Conversely, suppose $m \in UC(G)^*$ is left invariant. We want to show that $m(f * \phi) = m(\phi)$ for all $f \in P(G)$ and all $\phi \in UC(G)$. Fix $\phi \in UC(G)$ and define $H : L^1(G) \to \mathbb{C}$ by $\psi \mapsto m(\psi * \phi)$. Recall that if $\psi \in L^1(G)$ and $\phi \in UC(G)$ then $\psi * \phi \in UC(G)$, so H is well-defined. It is clear that $H \in L^1(G)^*$, so there exists some $\varphi \in L^\infty(G)$ such that $H(\psi) = \int_G \psi(g)\varphi(g)\, dg$. Since m is left-invariant, if we fix $g_0 \in G$ we have that $m(\psi * \phi) = m(\delta_{g_0} * \psi * \phi)$. However, using the definition of H,

$$m(\psi * \phi) = \int_G \psi(g)\varphi(g)\, dg$$

and

$$m(\delta_{g_0} * \psi * \phi) = \int_G (\delta_{g_0} * \psi)(g)\varphi(g)\, dg = \int_G \psi(g)(\delta_{g_0^{-1}} * \varphi)(g)\, dg$$

(where in the last step we use the left-invariance of the Haar measure). Since the two integrals are equal for all $\psi \in L^1(G)$ it follows that $\varphi(g) = (\delta_{g_0^{-1}} * \varphi)(g) = \varphi(g_0\, g)$. But g_0 was an arbitrary element of G, so in fact φ must be constant. Let $\varphi = c_0 \in \mathbb{C}$. Then $H(\psi) = c_0 \int_G \psi(g)\, dg$ for all $\psi \in L^1(G)$. In particular, for all $f \in P(G)$ we have that $\int_G f(g)\, dg = 1$, so $H(f) = c_0$.

Let $(e_\alpha)_\alpha \subset P(G)$ be a net such that $e_\alpha * \phi \to \phi$ for every $\phi \in UC(G)$ (for the existence of such a net see comment following Theorem 2.7). Then $m(e_\alpha * \phi) = c_0$. Hence for $f \in P(G)$ we have $m(f * \phi) = c_0 = \lim m(e_\alpha * \phi) = m(\phi)$ (since m is continuous and $e_\alpha * \phi \to \phi$). So if $\phi \in UC(G)$ is fixed, we have shown that $m(f * \phi) = m(\phi)$ for all $f \in P(G)$. Therefore, m is topologically invariant. $\square$

When we proved in the above theorem that every topological left invariant mean is left invariant we did not use the fact that the mean m was on the space $UC(G)$; in fact, the same proof can be used if m is a topologically left invariant mean on $L^\infty(G)$ or on $C_b(G)$. The converse does not hold in general; that is, not every left invariant mean is topologically left invariant. We denote by G_d the group obtained by equipping G with the discrete topology. In [14] it is proven that if G is a locally compact group such that G_d is also amenable, then the set of topologically invariant means is equal to the set of invariant means if and only if G itself is discrete (see [14], Theorem 7.21). We use the fact that every left invariant mean on $UC(G)$ is topologically left invariant to prove the following:

Theorem 2.10. *A locally compact topological group G is amenable if and only if there is a left invariant mean on $UC(G)$.*

Proof. If G is amenable, then there is a left invariant mean m on $L^\infty(G)$, and we can restrict m to a left invariant mean on $UC(G)$. Hence only the other direction needs proof.

Let m be a left invariant mean on $UC(G)$. Then by the previous theorem, m is also topologically left invariant. By Theorem 2.7 we can find a bounded approximate identity for $L^1(G)$ with elements in $P(G)$, and build an ultrafilter U on the index set of this bounded approximate identity such that it dominates the order filter. Let (e_α) be the net we obtain. Define $n \in L^\infty(G)^*$ by $n(\phi) = \lim_U m(e_\alpha * \phi * e_\alpha)$. Note that n is well-defined (since, by Theorem 2.8, $a * \phi * b \in UC(G)$ for all $a, b \in P(G)$ and $\phi \in L^\infty(G)$). Moreover, $n \geq 0$ since $m \geq 0$ and $n(1) = 1$, so n is a mean on $L^\infty(G)$. Finally, we just need to check that n is left-invariant. By the comments following Theorem 2.9, it is enough to check that n is topologically left invariant. For $f \in P(G)$ and $\phi \in L^\infty(G)$ we have:

$$
\begin{aligned}
n(f * \phi) &= \lim_U m(e_\alpha * f * \phi * e_\alpha) && \text{(by definition)} \\
&= \lim_U m(f * e_\alpha * \phi * e_\alpha) && \text{(since } f \in P(G) \text{ and } (e_\alpha)_\alpha \text{ is an} \\
& && \qquad \text{approximate identity for } P(G)) \\
&= \lim_U m(e_\alpha * \phi * e_\alpha) && \text{(since } m \text{ is topologically left invariant)} \\
&= n(\phi) && \text{(by definition of } n)
\end{aligned}
$$

Thus n is topologically left invariant, and hence left invariant. Therefore, G is amenable. $\square$

Using the above theorem we can also show that G is amenable if and only if there is a left-invariant mean on $C_b(G)$. This follows since $UC(G)$ is a subspace of $C_b(G)$, which in turn is a subspace of $L^\infty(G)$. Hence, if G is amenable, we can restrict the mean on $L^\infty(G)$ to a mean on $C_b(G)$. Conversely, if there is a mean on $C_b(G)$, then we can restrict it to a mean on $UC(G)$, so by the above theorem we get that G is amenable.

Earlier in this chapter we mentioned without proof some of the stability results for amenable groups. However, the fact that amenability is preserved by homomorphisms is of particular interest to us, and so we present this result below. We will later show that similar results hold for amenability of Banach algebras and for total reductivity.

Theorem 2.11. *Let G, H be locally compact groups. If G is amenable and $\phi : G \to H$ is a continuous homomorphism with dense range, then H is amenable as well.*

Proof. Let m be a left-invariant mean on $C_b(G)$ (such a mean exists since G is amenable). Note that if $\xi : H \to \mathbb{C}$ is continuous and bounded, then $\xi \circ \phi : G \to \mathbb{C}$ is also continuous and bounded. So we can define $n : C_b(H) \to \mathbb{C}$ by $n(\xi) = m(\xi \circ \phi)$. Then n is linear and $n(\mathbf{1}) = m(\mathbf{1}) = 1$. Moreover $|n(\xi)| = |m(\xi \circ \phi)| \leq \|\xi \circ \phi\| \leq \|\xi\|$, so $\|n\| \leq 1$. Hence it follows that n is a mean on $C_b(H)$.

We claim that n is also left-invariant. Consider $h_0 \in \operatorname{ran}(\phi)$; let $g_0 \in G$ be such that $\phi(g_0) = h_0$. Then for any $\xi \in C_b(H)$ we have

$$\begin{aligned}
((\delta_{h_0} * \xi) \circ \phi)(g) &= (\delta_{h_0} * \xi)(\phi(g)) \\
&= \xi(h_0^{-1}\phi(g)) \\
&= \xi(\phi(g_0)^{-1}\phi(g)) \\
&= \xi(\phi(g_0^{-1}g)) && \text{(since } \phi \text{ is a homomorphism)} \\
&= (\delta_{g_0} * (\xi \circ \phi))(g).
\end{aligned}$$

We use this fact in the following calculation

$$\begin{aligned}
n(\delta_{h_0} * \xi) &= m((\delta_{h_0} * \xi) \circ \phi) && \text{(by definition of } n) \\
&= m(\delta_{g_0} * (\xi \circ \phi)) && \text{(as shown above)} \\
&= m(\xi \circ \phi) && \text{(since } m \text{ is left-invariant)} \\
&= n(\xi)
\end{aligned}$$

Therefore, if $h_0 \in \operatorname{ran}(\phi)$, then $n(\delta_{h_0} * \xi) = n(\xi)$.

Now for any $h \in H$ we can find $(h_\alpha)_\alpha \subset \operatorname{ran}(\phi)$ converging to h (since the range of ϕ is dense in H). So $n(\delta_h * \xi) = \lim_\alpha n(\delta_{h_\alpha} * \xi) = \lim_\alpha n(\xi) = n(\xi)$. Hence in fact $n(\delta_h * \xi) = n(\xi)$ for any $\xi \in C_b(H)$ and $h \in H$. Therefore, n is a left-invariant mean on $C_b(H)$, and so, by the comments following Theorem 2.10, H is amenable. □

Let $\mathcal{M}$ be the set of means on $UC(G)$. Then $\mathcal{M}$ is convex and weak*-compact. We can define an action of G on $\mathcal{M}$ by $(g \cdot m)(\phi) = m(\delta_g * \phi)$ for $g \in G$, $m \in \mathcal{M}$ and $\phi \in UC(G)$. If m_0 is a fixed point of this action, ie. $g \cdot m_0 = m_0$ for all $g \in G$, then m_0 is a left-invariant mean. It follows that a fixed point exists if and only if G is amenable. If G is amenable the existence of a fixed point can be deduced in a more general context, as shown below. We say that an action of G on a set K is affine if $g \cdot (tx + (1-t)y) = t(g \cdot x) + (1-t)(g \cdot y)$ for $g \in G, x, y \in K, t \in [0,1]$.

Theorem 2.12. *[Day's Fixed Point Theorem] Let G be an amenable, locally compact group. Suppose E is a locally convex space, and $K \subset E$ is convex and compact. If G acts affinely on K, and the function $(g,k) \mapsto g \cdot k$ from $G \times K$ to K is separately continuous (continuous if either g or k is fixed), then there exists a $k_0 \in K$ such that $g \cdot k_0 = k_0$ for all $g \in G$.*

Proof. Let m be a left-invariant mean on $C_b(G)$ (m exists since G is amenable). Let $\mathcal{A}$ be the set of all affine, continuous functions on K. Fix $x_0 \in K$, and for each $\phi \in \mathcal{A}$ define $\psi_\phi : G \to \mathbb{C}$ by $g \mapsto \phi(g \cdot x_0)$. Note that $\psi_\phi \in C_b(G)$, since ϕ and $g \mapsto g \cdot x_0$ are continuous, and ϕ is bounded.

For each ϕ in $\mathcal{A}$ define ϕ_{g_0} by $k \mapsto \phi(g_0 \cdot k)$. Then ϕ_{g_0} is also in $\mathcal{A}$. We want to show that $m(\psi_\phi) = m(\psi_{\phi_{g_0}})$. By definition, $\psi_{\phi_{g_0}}(g) = \phi_{g_0}(g \cdot x_0) = \phi(g_0 \cdot g \cdot x_0)$. On the other hand, note that $(\delta_{g_0^{-1}} * \psi_\phi)(g) = \psi_\phi(g_0 g) = \phi(g_0 g \cdot x_0)$, so $\delta_{g_0^{-1}} * \psi_\phi = \psi_{\phi_{g_0}}$. Since m is left invariant, we have $m(\psi_\phi) = m(\delta_{g_0^{-1}} * \psi_\phi) = m(\psi_{\phi_{g_0}})$.

From a previous discussion, we know we can write m as the limit of a net $(m_\alpha)_\alpha$, where each m_α is an affine combination of means m_{g_i}, where $m_{g_i}(\psi) = \psi(g_i)$. For a fixed α, suppose $m_\alpha = \sum_{i=1}^n t_i m_{g_i}$. Then $m_\alpha(\psi_\phi) = \sum_{i=1}^n t_i \psi_\phi(g_i) = \phi(\sum_{i=1}^n t_i \cdot g_i \cdot x_0)$ (using the definition of ψ_ϕ and the fact that ϕ is an affine function). Let $k_\alpha = \sum_{i=1}^n t_i(g_i \cdot x_0)$; so $m_\alpha(\psi_\phi) = \phi(k_\alpha)$, where k_α is independent of ϕ. We have constructed a net $(k_\alpha)_\alpha$ in K.

Since K is compact we can assume without loss of generality that k_α converges to some $k_0 \in K$.

We will show that k_0 is our desired fixed point. First note that for any ϕ in $\mathcal{A}$, $\phi(k_0) = \lim_\alpha \phi(k_\alpha) = \lim_\alpha m_\alpha(\psi_\phi) = m(\psi_\phi)$ (using the continuity of ϕ). Now consider any $\mu \in E^*$. Then $\mu|_K$ is a continuous, affine function on K. For any $g \in G$ we have that $\mu(k_0) = m(\psi_\mu) = m(\psi_{\mu_g}) = \mu_g(k_0) = \mu(g \cdot k_0)$. Thus $\mu(k_0) = \mu(g \cdot k_0)$ for any $\mu \in E^*$ and any $g \in G$. Since E^* separates points of E it follows that $k_0 = g \cdot k_0$ for any $g \in G$, so k_0 is a fixed point of the action of G on K. □

In particular, this theorem can be applied when E is the dual of a vector space. It is in fact used in this manner later, in the proof of Johnson's theorem. Though this is not used later in this thesis, it should be noted that the hypothesis of the above theorem can be relaxed to require the existence of just one $x_0 \in K$ such that $g \mapsto g \cdot x_0$ is continuous; some of the useful consequences of this modification were pointed out by Anthony Lau at the 2006 Istanbul International Abstract Harmonic Analysis Conference.

2.2 Representation of Groups

In [12], Kadison examines representations of groups and algebras and the occurrence of certain similarity conditions. In particular, for group representations, he concerns himself with the question of when a similarity matrix can be applied to the representation in such a way that the operators in the range of the representation become unitary. Kadison also examines some of the connections between the similarity question for groups and the one for algebras.

Definition 2.13. *Let G be a locally compact group, and let E be a Banach space. A* ***representation*** *of G on E is a group homomorphism π from G into the invertible bounded operators on E which is continuous with respect to the given topology on G and the weak operator topology on $\mathcal{B}(E)$.*

For a locally compact group G we can define the function $\lambda : g \mapsto \lambda_g$, where $\lambda_g(f) = \delta_g * f$ for $f \in L^2(G)$. This is a representation of G on $L^2(G)$; it is called the left regular representation of G, and occurs frequently in the literature. Note moreover that λ is not continuous with respect to the norm topology on $\mathcal{B}(E)$ unless G is discrete. Hence the norm topology is considered too restrictive to be used in the definition of continuity given above. On the other hand, as shown by the theorem below, the weak operator topology could be replaced by the strong operator topology without affecting the set of homomorphisms under consideration.

Theorem 2.14. *Let G be a locally compact group, let E be a Banach space, and let $\pi : G \to \mathcal{B}(E)$ be a representation of G on E. Then π is continuous with respect to the given topology on G and the strong operator topology.*

Proof. Denote by e_G be the identity element of G. Let K be a compact neighbourhood of e_G, and choose U a symmetric neighbourhood of e_G such that $UU \subset K$. Since K is compact and π is a group representation, $\pi(K)$ is compact in the weak operator topology. It follows that if we fix $v_0 \in E$ then $\{\pi(g)v_0 : g \in K\}$ is compact in the weak topology, and hence, by the Uniform Boundedness Principle, it is bounded in the norm topology. A second application of the Uniform Boundedness Principle then gives us that $\{\|\pi(g)\| : g \in K\}$ is bounded, say by a constant C.

Define the set $F = \{v \in E : g \mapsto \pi(g)v$ is continuous with respect to the norm topology on $E\}$. In order to conclude that π is continuous with respect to the strong operator topology, we need to show that $F = E$.

First we shall prove that F is closed in the norm topology. Suppose $\{u_n\}_n$ is a sequence in F converging in the norm topology to some $u \in E$. Since π is a homomorphism, in order to conclude that $u \in F$ it is enough to show that whenever $\{e_\beta\}_\beta$ is a net converging to e_G in G we have $\pi(e_\beta)u \to \pi(e_G)u = u$. In addition, since K is a neighbourhood of e_G, we can assume without loss of generality that $\{e_\beta\}_\beta \subset K$. Given $\epsilon > 0$ choose n_0 such that $\|u_{n_0} - u\| < \frac{\epsilon}{2(C+1)}$. Since $u_{n_0} \in F$, by definition $\pi(e_\beta)u_{n_0} \to u_{n_0}$, and so we can find β_0 such that $\|\pi(e_\beta)u_{n_0} - u_{n_0}\| < \frac{\epsilon}{2}$ for $\beta \geq \beta_0$. Thus for $\beta \geq \beta_0$ we have

$$\begin{aligned}
\|\pi(e_\beta)u - u\| &\leq \|\pi(e_\beta)u - \pi(e_\beta)u_{n_0}\| + \|\pi(e_\beta)u_{n_0} - u_{n_0}\| + \|u_{n_0} - u\| \\
&\leq C\|u_{n_0} - u\| + \tfrac{\epsilon}{2} + \|u_{n_0} - u\| \\
&\qquad \text{(since } \|\pi(g)\| \leq C \text{ for all } g \in K) \\
&\leq C\tfrac{\epsilon}{2(C+1)} + \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2(C+1)} \\
&= \epsilon.
\end{aligned}$$

Hence $\pi(e_\beta)u \to u$ as desired, so $u \in F$. Therefore, F is closed, and in particular it is weakly closed.

Let $(\psi_\alpha)_\alpha$ be a bounded approximate identity for $L^1(G)$, where each ψ_α is a continuous function with support in U, as described in Theorem 2.7. Fix $v_0 \in E$. We will construct a net $(v_\alpha)_\alpha \subset E$ such that $v_\alpha \in F$ and $v_\alpha \overset{wk}{\to} v_0$. This will allow us to conclude that $v_0 \in F$. Let $v_\alpha = \int_G \psi_\alpha(g)[\pi(g)v_0]\,dg$. Recall that ψ_α is a function in $L^1(G)$ and was defined to have support in the compact set K, so since $[\pi(g)v_0]$ is bounded the integral can be calculated (for an overview of vector-valued integrals see [22], p. 12). We will show that $v_\alpha \in F$, and that $v_\alpha \overset{wk}{\to} v_0$. By definition, $v_\alpha \in F$ for a fixed α if $g \mapsto \pi(g)v_\alpha$ is continuous with respect to the norm topology on E. As above, it is enough to show that if $(e_\beta)_\beta \subset U$ is a net such that $e_\beta \to e_G$ then $\pi(e_\beta)v_\alpha \to v_\alpha$. We have

$$\begin{aligned}\pi(e_\beta)v_\alpha &= \pi(e_\beta)(\textstyle\int_G \psi_\alpha(g)\pi(g)v_0)\,dg\\ &= \textstyle\int_G \psi_\alpha(g)\pi(e_\beta g)v_0\,dg\\ &= \textstyle\int_G \psi_\alpha(e_\beta^{-1}g)\pi(g)v_0\,dg.\end{aligned}$$

Hence

$$\begin{aligned}\|\pi(e_\beta)v_\alpha - v_\alpha\| &= \|\textstyle\int_G[\psi_\alpha(e_\beta^{-1}g)\pi(g)v_0 - \psi_\alpha(g)\pi(g)v_0]\,dg\|\\ &\le \|\pi(g)v_0\|\|\psi_\alpha(e_\beta^{-1}g) - \psi_\alpha(g)\|_1.\end{aligned}$$

But $\|\psi_\alpha(e_\beta^{-1}g) - \psi_\alpha(g)\|_1 \to 0$ as $e_\beta \to e_G$, so since $\{\|\pi(g)\| : g \in K\}$ is bounded and $\|v_0\|$ is fixed, it follows that $\|\pi(e_\beta)v_\alpha - v_\alpha\| \to 0$. Therefore, $v_\alpha \in F$ as claimed.

Now we need to show that $v_\alpha \overset{wk}{\to} v_0$. Fix $\phi \in E^*$. Then $\phi(v_\alpha) = \int_G \psi_\alpha(g)\phi(\pi(g)v_0)\,dg$. For a fixed α we have that $\phi(v_0) = \int_G \psi_\alpha(g)\phi(v_0)\,dg$ (since $\phi(v_0)$ does not depend on g, and $\psi_\alpha \in P(G)$). Hence $\|\phi(v_\alpha) - \phi(v_0)\| = \|\int_G \psi_\alpha(g)[\phi(\pi(g)v_0) - \phi(\pi(e_G)v_0)]\,dg\|$. But ϕ is continuous with respect to the weak operator topology; moreover, $\{\alpha\}$ is a neighbourhood basis of e_G and the support of ψ_α is contained in $\alpha \subset K$, where K is compact. Hence for any $\epsilon > 0$ we can find a neighbourhood α_0 of e_G such that $\|\phi(\pi(g)v_0) - \phi(\pi(e_G)v_0)\| < \epsilon$ for $g \in \alpha_0$. Therefore, $\phi(\pi(g)v_0) \to \phi(v_0)$.

Thus $v_\alpha \overset{wk}{\to} v_0$. Since $v_\alpha \in F$ for each α and F is closed in the weak topology, we can conclude that $v_0 \in F$. But $v_0 \in E$ was arbitrary, so $E \subset F$. Therefore, $g \mapsto \pi(g)v$ is continuous for each $v \in E$, and hence π is continuous with respect to the strong operator topology. □

Definition 2.15. *Let G be a locally compact group, and let E be a Banach space. Two representations π_1 and π_2 of G on E are* ***similar*** *if there is an invertible operator $T \in \mathcal{L}(E)$ such that $\pi_1(g) = T\pi_2(g)T^{-1}$ for all $g \in G$.*

A representation π of G on E is ***uniformly bounded*** *if* $\sup_{g\in G}\|\pi(g)\| < \infty$.
A representation π of G on a Hilbert space $\mathcal{H}$ is ***unitary*** *if $\pi(g)$ is unitary for each $g \in G$.*

An example of a continuous unitary representation is the left regular representation defined at the beginning of this section.

It is clear that every representation similar to a unitary representation is uniformly bounded, leading us to the question of what are the necessary conditions for all the (continuous) uniformly bounded representations of a group G to be similar to unitary representations. A history of this question and the various related results can be found in the Introduction of [15]. In particular, the following theorem was proven independently by Dixmier and Day in 1950.

Theorem 2.16. *Let G be an amenable locally compact group and let $\pi : G \to \mathcal{H}$ be a uniformly bounded representation on some Hilbert space $\mathcal{H}$. Then there exists a similarity matrix T such that $T^{-1}\pi T$ is a unitary representation. Moreover, if $C = \sup\limits_{g\in G}\|\pi(g)\|$, then T can be chosen such that $\|T\|\|T^{-1}\| \leq C^2$.*

Proof. For $u, v \in \mathcal{H}$ define a function on G by $\phi_{uv}(g) = \langle \pi(g^{-1})u \,|\, \pi(g^{-1})v\rangle$.

First we will show that ϕ_{uv} is continuous. Consider $g_\alpha \to g$ a net in G. Then, since π is continuous with respect to the topology on G and the strong operator topology on $\mathcal{B}(\mathcal{H})$, it follows that $\|\pi({g_\alpha}^{-1})w - \pi(g^{-1})w\| \to 0$ for any $w \in \mathcal{H}$. Hence we have that

$$\begin{aligned}
|\phi_{uv}(g_\alpha) - \phi_{uv}(g)| &= |\langle \pi({g_\alpha}^{-1})u \,|\, \pi({g_\alpha}^{-1})v\rangle - \langle \pi(g^{-1})u \,|\, \pi(g^{-1})v\rangle| \\
&= |\langle \pi({g_\alpha}^{-1})u \,|\, \pi({g_\alpha}^{-1})v\rangle - \langle \pi({g_\alpha}^{-1})u \,|\, \pi(g^{-1})v\rangle + \\
&\qquad \langle \pi({g_\alpha}^{-1})u \,|\, \pi(g^{-1})v\rangle - \langle \pi(g^{-1})u \,|\, \pi(g^{-1})v\rangle| \\
&\leq |\langle \pi({g_\alpha}^{-1})u \,|\, \pi({g_\alpha}^{-1})v - \pi(g^{-1})v\rangle| + \\
&\qquad |\langle \pi({g_\alpha}^{-1})u - \pi(g^{-1})u \,|\, \pi(g^{-1})v\rangle| \\
&\leq \|\pi({g_\alpha}^{-1})u\|\|\pi({g_\alpha}^{-1})v - \pi(g^{-1})v\| + \\
&\qquad \|\pi({g_\alpha}^{-1})u - \pi(g^{-1})u\|\|\pi(g^{-1})v\| \text{ (by Cauchy-Schwarz)} \\
&\leq C\|u\|\|\pi({g_\alpha}^{-1})v - \pi(g^{-1})v\| + \|\pi({g_\alpha}^{-1})u - \pi(g^{-1})u\|C\|v\|
\end{aligned}$$

But $\|u\|$ and $\|v\|$ are constants, $\|\pi({g_\alpha}^{-1})v - \pi(g^{-1})v\| \to 0$, and likewise $\|\pi({g_\alpha}^{-1})u - \pi(g^{-1})u\| \to 0$. It follows that $|\phi_{uv}(g_\alpha) - \phi_{uv}(g)| \to 0$, and hence ϕ_{uv} is continuous.

Moreover, for a fixed $g \in G$ we also have that

$$\begin{aligned} |\phi_{uv}(g)| &= |\langle \pi(g^{-1})u \,|\, \pi(g^{-1})v \rangle| \\ &\leq \|\pi(g^{-1})u\| \|\pi(g^{-1})v\| \quad \text{(by Cauchy-Schwarz)} \\ &\leq C\|u\| C\|v\| \quad \text{(by definition of } C\text{)} \end{aligned}$$

so, since $\|u\|$ and $\|v\|$ are constant for this function, ϕ_{uv} is bounded by $C^2\|u\|\|v\|$. Hence we have shown that ϕ_{uv} is in $C_b(G)$.

Let m be a left invariant mean on $C_b(G)$. Define $[u, v] = m(\phi_{uv})$. Since m is linear and $\langle \cdot \,|\, \cdot \rangle$ is sesquilinear, it follows that $[\cdot, \cdot]$ is a sesquilinear form. Moreover, m is a positive functional, and ϕ_{uu} is a positive function (once again because $\langle \cdot \,|\, \cdot \rangle$ is an inner product), so it also follows that $[\cdot, \cdot]$ is positive semidefinite. So we can define a seminorm on $\mathcal{H}$ given by $|||u||| = [u, u]^{1/2}$.

Next we show that $|||\cdot|||$ is equivalent to the usual norm on $\mathcal{H}$. Consider a fixed $u \in \mathcal{H}$. First note that

$$\begin{aligned} |||u|||^2 &= m(\phi_{uu}) \\ &\leq \|m\|\|\phi_{uu}\| \\ &= \sup_{g \in G} |\langle \pi(g^{-1})u \,|\, \pi(g^{-1})u \rangle| \quad (\text{since } \|m\| = 1) \\ &= \sup_{g \in G} \|\pi(g^{-1})u\|^2 \\ &\leq C^2\|u\|^2 \end{aligned}$$

and hence $|||u||| \leq C\|u\|$. On the other hand, for any $g \in G$ we have

$$\|u\| = \|\pi(g)\pi(g^{-1})u\| \leq C\|\pi(g^{-1})u\|,$$

so $\frac{1}{C^2}\|u\|^2 \leq \|\pi(g^{-1})u\|^2 = \phi_{uu}(g)\, \forall g \in G$. Since m is positive it follows that $m(\frac{1}{C^2}\|u\|^2) \leq m(\phi_{uu})$. But $m(\mathbf{1}) = 1$ and m is linear, so $m(\frac{1}{C^2}\|u\|^2) = \frac{1}{C^2}\|u\|^2$; and $m(\phi_{uu}) = |||u|||^2$ by definition. Thus $\frac{1}{C}\|u\| \leq |||u|||$.

Therefore we have shown that $\frac{1}{C}\|u\| \leq |||u||| \leq C\|u\|$ for any $u \in \mathcal{H}$, so the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. Using the Riesz Representation Theorem and the fact that the inner products are equivalent, we can find an invertible operator S such that for all $u, v \in \mathcal{H}$ we have $\langle Su|v\rangle = [u, v]$ (whence it follows that $[S^{-1}u, v] = \langle u|v\rangle$). The following calculation

$$\|Su\|^2 = \langle Su|Su\rangle = [u, Su] \leq |||u||| |||Su||| \leq C\|u\|C\|Su\|$$

shows that $\|S\| \leq C^2$, and similarly we get $\|S^{-1}\| \leq C^2$.

S is a positive operator, since for $u \neq 0$ we have $\langle Su|u\rangle = [u,u] > 0$. Hence there exists an invertible, self-adjoint operator T such that $T = S^{1/2}$. Note that $\|T\| = \|S\|^{1/2}$ and $\|T^{-1}\| = \|S^{-1}\|^{1/2}$, so from $\|S\| \leq C^2$ and $\|S^{-1}\| \leq C^2$ we get $\|T\|\|T^{-1}\| \leq C^2$. Note that, for $Q \in \mathcal{B}(\mathcal{H})$, if we denote by $Q^{[*]}$ the adjoint with respect to $[\cdot,\cdot]$ and by Q^* the adjoint with respect to $\langle\cdot|\cdot\rangle$, then $Q^* = S^*Q^{[*]}(S^*)^{-1} = SQ^{[*]}S^{-1}$. Hence the adjoint of T is the same with respect to both inner products.

We will show that T is the similarity matrix which changes π into a unitary representation. Fix $h \in G$; we need to show that $T\pi(h)T^{-1}$ is a unitary operator. We have

$$\begin{aligned}\langle T\pi(h)T^{-1}x|T\pi(h)T^{-1}y\rangle &= \langle T^*T\pi(h)T^{-1}x|\pi(h)T^{-1}y\rangle\\ &= \langle S\pi(h)T^{-1}x|\pi(h)T^{-1}y\rangle\\ &= [\pi(h)T^{-1}x,\pi(h)T^{-1}y]\\ &= m(\phi_{\pi(h)T^{-1}x,\pi(h)T^{-1}y}).\end{aligned}$$

Since

$$\begin{aligned}\phi_{\pi(h)T^{-1}x,\pi(h)T^{-1}y}(g) &= \langle\pi(g^{-1})\pi(h)T^{-1}x|\pi(g^{-1})\pi(h)T^{-1}y\rangle\\ &= \langle\pi((h^{-1}g)^{-1})T^{-1}x|\pi((h^{-1}g)^{-1})T^{-1}y\rangle\end{aligned}$$

we get $\phi_{\pi(h)T^{-1}x,\pi(h)T^{-1}y} = \delta_h * \phi_{T^{-1}x,T^{-1}y}$. But m is a left invariant mean; so

$$\begin{aligned}m(\delta_h * \phi_{T^{-1}x,T^{-1}y}) &= m(\phi_{T^{-1}x,T^{-1}y})\\ &= [T^{-1}x,T^{-1}y]\\ &= [T^{-2}x,y]\\ &= [S^{-1}x,y]\\ &= \langle x|y\rangle\end{aligned}$$

Thus for every $h \in G$ and $x, y \in \mathcal{H}$ we have $\langle T\pi(h)T^{-1}x|T\pi(h)T^{-1}y\rangle = \langle x|y\rangle$, and hence $T\pi(h)T^{-1}$ is a unitary operator.

Therefore we have found an invertible operator T such that $T\pi(\cdot)T^{-1}$ is a unitary representation, and $\|T\|\|T^{-1}\| \leq C^2$, as desired. $\square$

If we drop the amenability requirement in the above theorem, then we can give an example of a group for which the result no longer holds. We denote by $\mathbb{F}_\infty$ the free group on countably many generators. Using the construction described in [15] we can exhibit a representation of $G = \mathbb{F}_\infty$ on $l_2(G) \oplus l_2(G)$ which is not unitarizable.

Let $U = \{u_n : n \in \mathbb{N}\}$ be the set of generators of G. Denote by $|g|$ the length of $g \in G$ as a reduced word. As usual, let δ_g be the function which evaluates to 1 at g and 0 everywhere else. Recall that δ_g is an orthonormal basis for $l_2(G)$. In the following we denote the empty word by v (usually we use e for the identity in G, but we will need e for the natural logarithm later in this proof).

For a fixed $g \in G$ we define $[\lambda_g(F)](t) = F(g^{-1}t)$ for $F \in l_2(G)$. Then we have $\sum \|F(g^{-1}t)\|^2 = \sum \|F(t)\|^2$, so $\lambda_g(F)$ is also in $l_2(G)$ and λ_g has norm 1. Also define a function ϕ_g on $l_2(G)$ by $[\phi_g(F)](t) = \sum\limits_{\substack{b \in U \cup U^{-1} \\ |g^{-1}t| > |g^{-1}tb|}} F(g^{-1}tb) - \sum\limits_{\substack{a \in U \cup U^{-1} \\ |t| > |ta|}} F(g^{-1}ta)$. Note that each sum in the definition of $[\phi_g(F)](t)$ has finitely many terms; in fact, if for $g \in G$ we denote by g_0 the first letter of g and by g_l the last, then the above function evaluates to

$$[\phi_g(F)](t) = \begin{cases} 0 & \text{if } t = g = v \\ F(g^{-1}g_0) & \text{if } t = v \text{ and } g \neq v \\ -F({t_l}^{-1}) & \text{if } t \neq v \text{ and } g = t \\ F(k^{-1}k_0) - F(k^{-1}{t_l}^{-1}) & \text{if } t \neq v \text{ and } g = tk \text{ for some } k \neq v \\ & \qquad \text{with } k_0 \neq {t_l}^{-1} \\ 0 & \text{otherwise} \end{cases}$$

In the above definition of ϕ_g, g is fixed; hence there can only be finitely many t for which there is a $k \in G$ such that $g = tk$ as a reduced word. The other values where $[\phi_g(F)](t)$ might be non-zero are $t = g$ when $g \neq v$ or $t = v$. So there are only finitely many t for which $[\phi_g(F)](t) \neq 0$. Therefore, $\phi_g(F) \in l_2(G)$.

In particular, consider $F = \delta_v$. Then clearly $-\delta_v({t_l}^{-1}) = 0$ for any $t \neq v$. If $|g| = 1$, then $\delta_v(g^{-1}g_0) = 1$ and there are no values of $t, k \neq v$ for which $g = tk$ (as a reduced word) so $[\phi_g(\delta_v)](t) = 0$ in all other cases; therefore, if $|g| = 1$ then $\phi_g(\delta_v) = \delta_v$. On the other hand, if $|g| \neq 1$ then $[\phi_g(\delta_v)](t) = 0$, hence $\phi_g(\delta_v) \subset \{\delta_v\}^{\perp}$.

Define the representation $\pi : g \mapsto \begin{bmatrix} \lambda_g & \phi_g \\ 0 & \lambda_g \end{bmatrix}$. First we show that $\pi(g)\pi(h) = \pi(gh)$ for $g, h \in G$, so π is indeed a homomorphism. Fix $g, h \in G$. It is straightforward to check that $\lambda_g\lambda_h = \lambda_{gh}$. Over the next couple of pages we will show that $\lambda_g\phi_h + \phi_g\lambda_h = \phi_{gh}$ for all $g, h \in G$. Consider $F \in l_2(G)$ and $t \in G$, and define $K(t) = F(h^{-1}t)$. Then

$[(\lambda_g\phi_h+\phi_g\lambda_h)(F)](t)=[\phi_hF](g^{-1}t)+[\phi_gK](t)$. But

$$(\phi_hF)(g^{-1}t)=\begin{cases}0 & \text{if } g^{-1}t=h=v\\ F(h^{-1}h_0) & \text{if } g^{-1}t=v \text{ and } h\neq v\\ -F({(g^{-1}t)_l}^{-1}) & \text{if } g^{-1}t\neq v \text{ and } h=g^{-1}t\\ F(k^{-1}k_0)-F(k^{-1}{(g^{-1}t)_l}^{-1}) & \text{if } g^{-1}t\neq v \text{ with } h=g^{-1}tk \text{ for some}\\ & \qquad k\neq v \text{ with } k_0^{-1}\neq(g^{-1}t)_l\\ 0 & \text{otherwise}\end{cases}$$

and

$$(\phi_gK)(t)=\begin{cases}0 & \text{if } t=g=v\\ F(h^{-1}g^{-1}g_0) & \text{if } t=v \text{ and } g\neq v\\ -F(h^{-1}{t_l}^{-1}) & \text{if } t\neq v \text{ and } g=t\\ F(h^{-1}k^{-1}k_0)-F(h^{-1}k^{-1}{t_l}^{-1}) & \text{if } t\neq v \text{ and } g=tk \text{ for some } k\neq v\\ & \qquad \text{with } k_0\neq {t_l}^{-1}\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand,

$$[\phi_{gh}(F)](t)=\begin{cases}0 & \text{if } t=gh=v\\ F(h^{-1}g^{-1}(gh)_0) & \text{if } t=v \text{ and } gh\neq v\\ -F({t_l}^{-1}) & \text{if } t\neq v \text{ and } gh=t\\ F(k^{-1}k_0)-F(k^{-1}{t_l}^{-1}) & \text{if } t\neq v \text{ and } gh=tk \text{ for some } k\neq v\\ & \qquad \text{with } k_0\neq {t_l}^{-1}\\ 0 & \text{otherwise.}\end{cases}$$

We want to compare the function defined above with $[\phi_hF](g^{-1}t)+[\phi_gK](t)$, so we need to consider all possible cases for values of t, g and h. For ease of calculation, we will use the cases in the definition of ϕ_gK, and consider subcases as necessary so we can evaluate $\phi_{gh}F$ and ϕ_hF.

- Suppose $t=g=v$, so $[\phi_gK](t)=0$.
    - If $h=v$, then $[\phi_hF](g^{-1}t)=0$ and $[\phi_{gh}F](t)=0$.

– If $h \neq v$, then $[\phi_h F](g^{-1}t) = F(h^{-1}h_0)$, and
$[\phi_{gh}F](t) = F(h^{-1}g^{-1}(gh)_0) = F(h^{-1}h_0)$.

Therefore, for any value of h, $[\phi_h F](g^{-1}t) + [\phi_g K](t) = [\phi_{gh}F](t)$.

- Suppose $t = v$ and $g \neq v$. Then $g^{-1}t \neq v$, so we need to consider three cases:

– If $h = g^{-1}t = g^{-1}$, then
$[\phi_h F](g^{-1}t) = -F({(g^{-1}t)_l}^{-1}) = -F((g_0^{-1})^{-1}) = -F(g_0)$,
$[\phi_g K](t) = F(h^{-1}g^{-1}g_0) = F(g_0)$ and $[\phi_{gh}F](t) = 0$.

– If $h = g^{-1}tk = g^{-1}k$ for some $k \neq v$ with $k_0^{-1} \neq (g^{-1}t)_l$, then
$[\phi_h F](g^{-1}t) = F(k^{-1}k_0) - F(k^{-1}{(g^{-1}t)_l}^{-1}) = F(k^{-1}k_0) - F(k^{-1}g_0)$,
$[\phi_g K](t) = F(h^{-1}g^{-1}g_0) = F(k^{-1}g_0)$ and
$[\phi_{gh}F](t) = F((gh)^{-1}(gh_0) = F(k^{-1}k_0)$.

– If h has neither of the above two forms, then $[\phi_h F](g^{-1}t) = 0$,
$[\phi_g K](t) = F(h^{-1}g^{-1}g_0)$ and $[\phi_{gh}F](t) = F(h^{-1}g^{-1}g_0)$ (since $gh \neq v$).

Therefore, for any value of h, $[\phi_h F](g^{-1}t) + [\phi_g K](t) = [\phi_{gh}F](t)$.

- Suppose $t \neq v$ and $g = t$. Then $g^{-1}t = v$.

– If $h = v$ then $[\phi_h F](g^{-1}t) = 0$, $[\phi_g K](t) = -F(h^{-1}{t_l}^{-1}) = -F({t_l}^{-1})$ and
$[\phi_{gh}F](t) = -F({t_l}^{-1})$ (since $gh = t$).

– If $h \neq v$ then $[\phi_h F](g^{-1}t) = F(h^{-1}h_0)$, and $[\phi_g K](t) = -F(h^{-1}{t_l}^{-1})$. If $h_0 \neq {t_l}^{-1}$ we can use $k = h$ to get $gh = tk$, so
$[\phi_{gh}F](t) = F(k^{-1}k_0) - F(k^{-1}{t_l}^{-1}) = F(h^{-1}h_0) - F(h^{-1}{t_l}^{-1})$. On the other hand, if $h_0 = {t_l}^{-1}$, then $[\phi_{gh}F](t) = 0$; however, note that in this case we also have $h^{-1}h_0 = h^{-1}{t_l}^{-1}$, and so $[\phi_h F](g^{-1}t) = -[\phi_g K](t)$.

Therefore, for any value of h, $[\phi_h F](g^{-1}t) + [\phi_g K](t) = [\phi_{gh}F](t)$.

- Suppose $t \neq v$ and $g = tk$ for some $k \neq v$ with $k_0 \neq {t_l}^{-1}$. Then $g^{-1}t = k^{-1} \neq v$.

– If $h = g^{-1}t = k^{-1}$, then $[\phi_h F](g^{-1}t) = -F({(g^{-1}t)_l}^{-1}) = -F(k_0)$,
$[\phi_g K](t) = F(h^{-1}k^{-1}k_0) - F(h^{-1}k^{-1}{t_l}^{-1}) = F(k_0) - F({t_l}^{-1})$, and
$[\phi_{gh}F](t) = -F({t_l}^{-1})$ (since $gh = t$).

- If $h = g^{-1}tr = k^{-1}r$ for some $r \neq v$ with $r_0 \neq k_0$ then
$[\phi_h F](g^{-1}t) = F(k^{-1}k_0) - F(k^{-1}{(g^{-1}t)_l}^{-1}) = F(k^{-1}k_0) - F(k^{-1}k_0) = 0$, and
$[\phi_g K](t) = F(h^{-1}k^{-1}k_0) - F(h^{-1}k^{-1}{t_l}^{-1}) = F(r^{-1}k_0) - F(r^{-1}{t_l}^{-1})$.
If $r_0 \neq {t_l}^{-1}$, then $gh = tr$ where tr is a reduced word, and so
$[\phi_{gh}F](t) = F(r^{-1}r_0) - F(r^{-1}{t_l}^{-1})$; otherwise, $[\phi_{gh}F](t) = 0$, but also
$r^{-1}k_0 = r^{-1}{t_l}^{-1}$ and so $[\phi_g K](t) = 0$.
- If h has neither of the above forms then $[\phi_h F](g^{-1}t) = 0$,
$[\phi_g K](t) = F(h^{-1}k^{-1}k_0) - F(h^{-1}k^{-1}{t_l}^{-1})$ and
$[\phi_{gh}F](t) = F((kh)^{-1}k_0) - F((kh)^{-1}{t_l}^{-1})$ (since $gh = tkh$, and $(kh)_0 \neq {t_l}^{-1}$).

Therefore, for any value of h, $[\phi_h F](g^{-1}t) + [\phi_g K](t) = [\phi_{gh}F](t)$.

- Suppose $t \neq v$ and g has neither of the previous two forms. Then $[\phi_g K](t) = 0$.
  - If $h = g^{-1}t$ then $[\phi_h F](g^{-1}t) = -F({(g^{-1}t)_l}^{-1}) = -F({t_l}^{-1})$, and
$[\phi_{gh}F](t) = -F({t_l}^{-1})$ (since $gh = t$).
  - If $h = g^{-1}tk$ for some $k \neq v$ with $k_0 \neq {(g^{-1}t)_l}^{-1}$ then
$[\phi_h F](g^{-1}t) = F(k^{-1}k_0) - F(k^{-1}{(g^{-1}t)_l}^{-1}) = F(k^{-1}k_0) - F(k^{-1}{t_l}^{-1})$ and
$[\phi_{gh}F](t) = F(k^{-1}k_0) - F(k^{-1}{t_l}^{-1})$ (since $gh = tk$, with $k_0 \neq {t_l}^{-1}$).
  - If h has neither of the above two forms then $[\phi_h F](g^{-1}t) = 0$ and $[\phi_{gh}F](t) = 0$.

  Therefore, for any value of h, $[\phi_h F](g^{-1}t) + [\phi_g K](t) = [\phi_{gh}F](t)$.

We conclude that $[\phi_{gh}F](t) = [\phi_h F](g^{-1}t) + [\phi_g K](t)$ for any values of g, h and t. Therefore, $\lambda_g \phi_h + \phi_g \lambda_h = \phi_{gh}$ for all $g, h \in G$.

Note that

$$\begin{aligned}\langle \pi(g)(0 \oplus \delta_v) \,|\, (\delta_v \oplus 0)\rangle &= \langle \lambda_g(0) + \phi_g(\delta_v) \,|\, \delta_v\rangle + \langle \lambda_g(\delta_v)|0\rangle \\ &= \begin{cases} 1 & \text{if } |g| = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{(using the earlier calculations for } F = \delta_v).\end{aligned}$$

Suppose π is unitarizable; so we can find a matrix S such that $\pi(g) = S^{-1}\rho(g)S$ for some unitary representation ρ. Let $F_1 = S(0 \oplus \delta_v)$ and $F_2 = (S^{-1})^*(\delta_v \oplus 0)$. So

$$\langle \rho(g)F_1|F_2\rangle = \langle \rho(g)S(0 \oplus \delta_v) \,|\, (S^{-1})^*(\delta_v \oplus 0)\rangle = \langle \pi(g)(0 \oplus \delta_v) \,|\, (\delta_v \oplus 0)\rangle$$

which evaluates to 1 if $|g| = 1$ and to 0 otherwise, as shown earlier.

Let $\alpha_n = [\rho(u_n) + \rho(u_n^{-1})]/2$ for $n \in \mathbb{N}$. Since ρ is unitary, $\|\alpha_n\| \leq 1$ and $\alpha_n^* = \alpha_n$. Define $R_n = \prod_{i=1}^{n}(I + \frac{i}{\sqrt{n}}\alpha_i)$. For each i we have

$$\begin{aligned}
\|(I + \tfrac{i}{\sqrt{n}}\alpha_i)\|^2 &= \|(I + \tfrac{i}{\sqrt{n}}\alpha_i)^*(I + \tfrac{i}{\sqrt{n}}\alpha_i)\| \\
&= \|(I - \tfrac{i}{\sqrt{n}}\alpha_i)(I + \tfrac{i}{\sqrt{n}}\alpha_i)\| \\
&= \|I + \tfrac{1}{n}\alpha_i^2\| \\
&\leq 1 + \tfrac{1}{n}
\end{aligned}$$

Hence $\|R_n\|^2 \leq \prod_{i=1}^{n}(1 + \frac{1}{n})^2 < \lim_{n\to\infty}(1 + \frac{1}{n})^{2n} = e^2$, so $\|R_n\| < e$.

We can multiply out the product used in the definition of R_n to obtain

$$R_n = I + \frac{i}{\sqrt{n}}(\alpha_1 + \ldots + \alpha_n) + (\frac{i}{\sqrt{n}})^2 \sum_{1\leq i<j\leq n} \alpha_i\alpha_j + \ldots + (\frac{i}{\sqrt{n}})^n \alpha_1 \ldots \alpha_n$$

Replacing α_n by its definition, and using the fact that ρ is a homomorphism, we can write R_n as a sum where each term is a scalar multiple of $\rho(g)$ for some $g \in G$. We are particularly interested in the terms which have $\rho(g)$ for $|g| = 1$. These terms are $\frac{i}{\sqrt{n}}(\alpha_1 + \ldots + \alpha_n) = \frac{i}{2\sqrt{n}} \sum_{i=1}^{n} \rho(u_i) + \rho(u_i^{-1})$. So we have

$$\begin{aligned}
\langle R_n F_1 | F_2 \rangle &= \langle S^{-1} R_n S(0 \oplus \delta_v) | (\delta_v \oplus 0) \rangle \\
&= \langle \left[\tfrac{i}{2\sqrt{n}} \sum_{i=1}^{n} \left(\pi(u_i) + \pi(u_i^{-1})\right) \right] (0 \oplus \delta_v) \,|\, (\delta_v \oplus 0) \rangle \\
&\qquad \text{(using the expansion of } R_n \text{ given above, and the fact that} \\
&\qquad \langle \pi(g)(0 \oplus \delta_v) \,|\, (\delta_v \oplus 0) \rangle \text{ is non-zero only when } |g| = 1) \\
&= \tfrac{i}{2\sqrt{n}}(2n) \\
&\qquad \text{(since } \langle \pi(g)(0 \oplus \delta_v) \,|\, (\delta_v \oplus 0) \rangle = 1 \text{ when } |g| = 1) \\
&= i\sqrt{n}
\end{aligned}$$

Since F_1 and F_2 are fixed, we conclude that $\|R_n\| \to \infty$, contradicting the earlier statement that $\|R_n\| < e$ for each n. Therefore, π is not unitarizable.

We have constructed a representation of $\mathbb{F}_\infty$ on $l_2(\mathbb{F}_\infty) \oplus l_2(\mathbb{F}_\infty)$ which is uniformly bounded but not unitarizable. Note that $\mathbb{F}_\infty$ can be embedded in $\mathbb{F}_n$ for any $n \in N$, $n \geq 2$;

so the above construction can be applied to show that we can find a non-unitarizable representation of $\mathbb{F}_n$.

Amenability is a sufficient condition for every bounded representation to be similar to a unitary representation, and we have seen above that we can find a non-amenable group such that this similarity condition no longer holds. In order to exhaust this line of inquiry we need to know if it is possible to find a non-amenable group for which all representations are unitarizable. This question, posed by Dixmier in 1950, remains open. Recent advances made in answering this question can be found in [16].

2.3 Banach Algebras

The concept of amenability for Banach algebras was introduced by Johnson. The theory was evolved as a consequence of the properties of the algebra $L^1(G)$ and used cohomology theory; in fact, while this section might seem unconnected to the earlier discussion for groups, the culminating result is that G is amenable if and only if $L^1(G)$ is amenable. Moreover, the concept can be applied to general Banach algebras with fruitful results.

Suppose $\mathcal{M}$ is a Banach space on which we define a module action of an algebra $\mathfrak{A}$. We say that $\mathcal{M}$ is a left Banach module of $\mathfrak{A}$ if there exists a k such that $\|a \cdot m\| \leq k\|a\|\|m\|$, and respectively a right Banach module of $\mathfrak{A}$ if there exists a t such that $\|m \cdot a\| \leq t\|m\|\|a\|$. If both the inequalities hold then $\mathcal{M}$ is a Banach-bimodule of $\mathfrak{A}$. Since the only modules we are concerned with in the following are Banach modules we will refer to them simply as $\mathfrak{A}$-modules, or just modules if the algebra $\mathfrak{A}$ can be inferred from context. Also, module will be understood to mean 'bimodule'; the explicit terms 'left module' and 'right module' will be used when such distinctions are necessary.

Note that if $\mathcal{M}$ is an $\mathfrak{A}$-module then we can define a representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{M})$ by $[\pi(a)](m) = a \cdot m$. On the other hand, if we have a representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{V})$ then we can define a module action of $\mathfrak{A}$ on $\mathcal{V}$ by $a \cdot v = [\pi(a)](v)$ for $a \in \mathfrak{A}$ and $v \in \mathcal{V}$. This equivalence between module actions and representations will be particularly relevant when we discuss operator algebras and invariant subspaces in later chapters.

Note also that if $\mathcal{M}$ is a bimodule of $\mathfrak{A}$ then we can make $\mathcal{M}^*$ into a bimodule by defining $(a \cdot \phi)(m) = \phi(m \cdot a)$ and $(\phi \cdot a)(m) = \phi(a \cdot m)$, where $a \in \mathfrak{A}$, $m \in \mathcal{M}$ and

$\phi \in \mathcal{M}^*$. It is straigtforward to check that this does indeed define a module action of $\mathfrak{A}$ on $\mathcal{M}^*$.

Definition 2.17. *A linear function $\rho : \mathfrak{A} \to \mathcal{M}$ is called a **derivation** if it satisfies $\rho(ab) = a \cdot \rho(b) + \rho(a) \cdot b$. Moreover, such a derivation is called **inner** if there is a fixed $\alpha \in \mathcal{M}$ such that $\rho(a) = a \cdot \alpha - \alpha \cdot a$ for all $a \in \mathfrak{A}$. The inner derivation ρ is often denoted by ad_α.*

Denote by $Z^1(\mathfrak{A}, \mathcal{M})$ the space of all continuous derivations from $\mathfrak{A}$ to $\mathcal{M}$ and by $B^1(\mathfrak{A}, \mathcal{M})$ the set of all inner derivations. We define $\mathcal{H}^1(\mathfrak{A}, M) = Z^1(\mathfrak{A}, \mathcal{M})/B^1(\mathfrak{A}, \mathcal{M})$, called the first Hochschild cohomology group of $\mathfrak{A}$ with coefficients in $\mathcal{M}$.

It is not always the case that all derivations are inner. Consider $\mathfrak{A}(\mathbb{D})$, the algebra of functions which are analytic inside the unit disc and continuous on $\overline{\mathbb{D}}$. We can make $\mathbb{C}$ into an $\mathfrak{A}(\mathbb{D})$ bimodule by defining $f \cdot c := f(0)c =: c \cdot f$. Note that since $f \cdot c = c \cdot f$, any inner derivation from $\mathfrak{A}(\mathbb{D})$ to $\mathbb{C}$ would in fact have to be 0. Define $D : \mathfrak{A}(\mathbb{D}) \to \mathbb{C}$ by $D(f) = f'(0)$. By properties of derivative, D is linear and satisfies

$$D(fg) = (fg)'(0) = f'(0)g(0) + f(0)g'(0) = f'(0) \cdot g + f \cdot g'(0) = D(f)g + fD(g)$$

Moreover, we can use the Cauchy Integral Formula to show that D is bounded. So D is a continuous derivation from $\mathfrak{A}(\mathbb{D})$ into $\mathbb{C}$, but $D \neq 0$ so D is not inner.

Definition 2.18. *A Banach algebra $\mathfrak{A}$ is **amenable** if $H^1(\mathfrak{A}, \mathfrak{B}^*) = 0$ for all Banach bimodules $\mathfrak{B}$ of $\mathfrak{A}$.*

Recall that amenability for groups was preserved by homomorphisms. The same result holds for Banach algebras, though of course the proof is different.

Theorem 2.19. *Let $\mathfrak{A}$ be an amenable Banach algebras. Suppose $\mathfrak{B}$ is a Banach algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism such that $\phi(\mathfrak{A})$ is dense in $\mathfrak{B}$. Then $\mathfrak{B}$ is also amenable.*

Proof. Suppose $\mathcal{M}^*$ is a $\mathfrak{B}$ dual bimodule, and $D : \mathfrak{B} \to \mathcal{M}^*$ is a continuous derivation. We need to find $\mu_0 \in \mathcal{M}^*$ such that $D(b) = b \cdot \mu_0 - \mu_0 \cdot b$ for all $b \in \mathfrak{B}$.

We can make $\mathcal{M}^*$ into an $\mathfrak{A}$ bimodule by defining $a \cdot \mu = \phi(a) \cdot \mu$ and $\mu \cdot a = \mu \cdot \phi(a)$ for any $a \in \mathfrak{A}$ and $\mu \in \mathcal{M}^*$. Since ϕ is continuous, it is easy to check that this definition satisfies all the requirements for $\mathcal{M}^*$ to be an $\mathfrak{A}$-bimodule. Moreover, we can define a derivation $D_{\mathfrak{A}} : \mathfrak{A} \to \mathcal{M}^*$ by $D_{\mathfrak{A}}(a) = D(\phi(a))$. But D and ϕ are both continuous, and hence so is $D_{\mathfrak{A}}$; moreover, since $\mathfrak{A}$ is amenable, $D_{\mathfrak{A}}$ must be inner. Thus there exists $\mu_0 \in \mathcal{M}^*$ such that $D_{\mathfrak{A}}(a) = a \cdot \mu_0 - \mu_0 \cdot a$ for any $a \in \mathfrak{A}$. But then, using the definition of $D_{\mathfrak{A}}$ and of the action of $\mathfrak{A}$ on $\mathcal{M}^*$ we see that $D(\phi(a)) = \phi(a) \cdot \mu_0 - \mu_0 \cdot \phi(a)$. Since $\phi(\mathfrak{A})$ is dense in $\mathfrak{B}$, by continuity of D and of the action, we get that $D(b) = b \cdot \mu_0 - \mu_0 \cdot b$ for any $b \in \mathfrak{B}$. Thus D is an inner derivation. Therefore, any derivation from $\mathfrak{B}$ to a dual space is inner, and hence $\mathfrak{B}$ is amenable. □

Our goal is to prove that the algebra $L^1(G)$ is amenable if and only if the group G is amenable. Building on our knowledge of amenable groups, this result will provide us with a large number of examples of amenable algebras.

For $\mathcal{C}^*$-algebras, amenability is equivalent to nuclearity. A $\mathcal{C}^*$-algebra $\mathfrak{A}$ is nuclear if for every $\mathcal{C}^*$-algebra $\mathfrak{B}$ there is exactly one $\mathcal{C}^*$-norm that can be defined on the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$. The fact that every amenable $\mathcal{C}^*$-algebra is nuclear was shown by Connes in [4], and the converse was established a few years later by Haagerup in [9]. From known results about nuclear $\mathcal{C}^*$-algebras it follows that every abelian $\mathcal{C}^*$-algebra is amenable.

Bounded approximate identities play a significant role in discussions of amenability for Banach algebras since it can be shown that any amenable algebra has a bounded approximate identity. A result about the stability of the amenability property says that a closed ideal of an amenable algebra is amenable if and only if it has a bounded approximate identity ([20], Theorem 2.2.1). Recall that any ideal in a $\mathcal{C}^*$-algebra has a bounded approximate identity; hence any closed ideal of an amenable $\mathcal{C}^*$-algebra is also amenable. A consequence of these observations is that we can restrict our discussion to Banach algebras which have a bounded approximate identity. This allows us to use the theorem stated below.

Theorem 2.20 (Cohen's Factorization Theorem)**.** *Let $\mathfrak{A}$ be a Banach algebra and E be a Banach left $\mathfrak{A}$-module. Suppose that there exists a bounded net $(e_\alpha)_\alpha$ in $\mathfrak{A}$ such that $e_\alpha \cdot x \to x$ for all $x \in E$. Then for every $z \in E$ and $\delta > 0$ we can find $a \in \mathfrak{A}$ and $y \in E$ such that $z = a \cdot y$ and $\|z - y\| < \delta$.*

Suppose we have $\mathfrak{A}$ and E as described in the above theorem. Since $e_\alpha \cdot x \to x$, it follows that $\phi \cdot e_\alpha \to \phi$ in the weak* topology on E^* for any $\phi \in E^*$. This is easy to see once we recall that the definition of the action for $\mathfrak{A}$ on E^* gives us that $[\phi \cdot e_\alpha](x) = \phi(e_\alpha \cdot x)$.

Theorem 2.21. *Let $\mathfrak{A}$ be a Banach algebra with a bounded right approximate identity, and let E be a Banach $\mathfrak{A}$-bimodule such that $\mathfrak{A} \cdot E = \{0\}$. Then $H^1(\mathfrak{A}, E^*) = \{0\}$.*
Note: The symmetric result "if $\mathfrak{A}$ has a bounded left approximate identity and $E \cdot \mathfrak{A} = \{0\}$ then $H^1(\mathfrak{A}, E^*) = \{0\}$" also holds.

Proof. Consider ρ a continuous derivation from $\mathfrak{A}$ to E^*. Let $(e_\alpha)_\alpha$ be a bounded right approximate identity for $\mathfrak{A}$. Then, since ρ is a bounded linear function, $\rho(e_\alpha)$ is bounded, so it has a weak*-limit point. By passing to a subnet as necessary, we can assume without loss of generality that $\rho(e_\alpha)$ is in fact convergent in the weak* topology to some $e \in E^*$.

For any $a \in \mathfrak{A}$ we have that $\rho(a) = \text{weak*-}\lim \rho(a \cdot e_\alpha)$ (since ρ is continuous). But $\rho(a \cdot e_\alpha) = \rho(a) \cdot e_\alpha + a \cdot \rho(e_\alpha)$ (by properties of derivations). Finally, since $\mathfrak{A} \cdot E = \{0\}$, it follows from the way E^* is defined as a bimodule that $E^* \cdot \mathfrak{A} = \{0\}$; hence $\rho(a) \cdot e_\alpha = 0$ for all $a \in \mathfrak{A}$ and α. So $\rho(a) = \text{weak*-}\lim a \cdot \rho(e_\alpha) = a \cdot e = a \cdot e - e \cdot a$. Hence we have in fact shown that ρ is an inner derivation. Therefore, $H^1(\mathfrak{A}, E^*) = \{0\}$. $\square$

Next we will show that we do not need to examine all the $\mathfrak{A}$-bimodules in order to show that $\mathfrak{A}$ is amenable. It is enough to consider modules of the type introduced below, whose advantages are explained in the remarks following the definition.

Definition 2.22. *If $\mathfrak{A}$ is a Banach algebra and E is a Banach $\mathfrak{A}$-bimodule, we say that E is* ***pseudo-unital*** *if $E = \{a \cdot x \cdot b : a, b \in \mathfrak{A}, x \in E\}$.*

Lemma 2.23. *Suppose that $\mathfrak{A}$ is a Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$, and $\mathcal{M}$ is a Banach $\mathfrak{A}$-bimodule. Then $\{a \cdot x : a \in \mathfrak{A}, x \in \mathcal{M}\}$ and $\{a \cdot x \cdot b : a, b \in \mathfrak{A}, x \in \mathcal{M}\}$ are closed subspaces of $\mathcal{M}$.*

Proof. Let $F_0 = \{a \cdot x : a \in \mathfrak{A}, x \in \mathcal{M}\}$ and $F = \overline{span\, F}$. If $y = a \cdot x$ is in F_0 then $e_\alpha a \to a$ (since e_α is an approximate identity), and so $e_\alpha \cdot (a \cdot x) = (e_\alpha a) \cdot x \to a \cdot x$. It then follows that $e_\alpha \cdot (\sum_{i=1}^{n} a_i \cdot x_i) \to (\sum_{i=1}^{n} a_i \cdot x_i)$ (by linearity of the module operation and since the sum has finitely many terms). Now fix $z \in F$; find a sequence $z_n \to z$ where $z_n \in span\, F$. By the definition of a Banach-bimodule there is a constant k such that for any α and n we have $\|e_\alpha \cdot (z - z_n)\| \leq k\|e_\alpha\|\|z - z_n\| \leq C\|z - z_n\|$ where C is a constant (the existence of C follows from the fact that $(e_\alpha)_\alpha$ is a bounded net). Hence, given $\epsilon > 0$

we can find N such that $\|z_N - z\| < \epsilon/3$ and $\|e_\alpha \cdot (z - z_N)\| < \epsilon/3$ for all α. But also $z_N \in span\, F$, so $e_\alpha z_N \to z_N$ from the earlier comments. It follows that we can find an α_0 such that $\|e_\alpha z_N - z_N\| < \epsilon/3$ for $\alpha \geq \alpha_0$. Combining these inequalities, for $\alpha \geq \alpha_0$ we get

$$\|e_\alpha \cdot z - z\| \leq \|e_\alpha \cdot z - e_\alpha \cdot z_N\| + \|e_\alpha \cdot z_N - z_N\| + \|z_N - z\| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

Therefore, we have shown that for any $z \in F$ we have $e_\alpha \cdot z \to z$. Hence, we can apply Cohen's Factorization Theorem (Theorem 2.20) to find $a \in \mathfrak{A}$ and $w \in F \subset \mathcal{M}$ such that $z = a \cdot w$. It follows that $z \in F_0$. Hence, since z was arbitrary, $F = F_0$. Therefore, $F_0 = \{a \cdot x : a \in \mathfrak{A}, x \in \mathcal{M}\}$ is a closed subspace of $\mathcal{M}$.

The proof that $\{a \cdot x \cdot b : a, b \in \mathfrak{A}, x \in \mathcal{M}\}$ is a closed subspace of $\mathcal{M}$ is similar to the above. □

Note that if $\mathfrak{A}$ is a Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$ and E is pseudo-unital, then $e_\alpha \cdot u \to u$ for any $u \in E$. This follows directly from the fact that any $u \in E$ can be written as $a \cdot v \cdot b$ for $a, b \in \mathfrak{A}$, and $e_\alpha a \to a$ for any $a \in \mathfrak{A}$. Therefore, as in the comment following Cohen's Factorization Theorem (Theorem 2.20), in this case we also have that $e_\alpha \cdot \phi \to \phi$ in the weak* topology on E^*.

Theorem 2.24. *Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity. If $H^1(\mathfrak{A}, F^*) = \{0\}$ whenever F is a pseudo-unital Banach $\mathfrak{A}$-bimodule, then $\mathfrak{A}$ is amenable.*

Proof. Let E be any Banach $\mathfrak{A}$-bimodule. We want to show that $H^1(\mathfrak{A}, E^*) = \{0\}$. Define $E_0 = \{a \cdot x \cdot b : a, b \in \mathfrak{A}, x \in E\}$ and $E_1 = \{a \cdot x : a \in \mathfrak{A}, x \in E\}$. By Lemma 2.23 E_0 and E_1 are closed subspaces of E. Moreover, E_0 is pseudo-unital.

Consider first a derivation D from $\mathfrak{A}$ into E_1^*. Define $D_0(a) = D(a)|_{E_0}$ for each $a \in \mathfrak{A}$. Then clearly D_0 is a derivation into E_0^*. Since E_0 is pseudo-unital, by hypothesis we can find $\phi \in E_0^*$ such that $D_0 = ad_\phi$. By the Hahn-Banach theorem we can find $\varphi \in E_1^*$ such that φ extends ϕ. Let $D_1 = D - ad_\varphi$. Then for any $a \in \mathfrak{A}$ we get

$$D_1(a)|_{E_0} = D(a)|_{E_0} - ad_\varphi(a)|_{E_0} = D_0(a) - ad_\phi(a) = 0$$

(since $D_0 = ad_\phi$ by definition, and φ extends ϕ). Therefore, $\operatorname{ran} D_1 \in E_0^\perp$. But $E_0^\perp \cong (E_1/E_0)^*$. Note that $(E_1/E_0) \cdot \mathfrak{A} = \{0\}$, so Theorem 2.21 gives us that $H^1(\mathfrak{A}, (E_1/E_0)^*) = \{0\}$. In particular, it follows that D_1, which is a derivation into $E_0^\perp$, is inner; hence, there exists a $\psi \in E_0^\perp$ such that $D_1 = ad_\psi$. Recall that $D = D_1 + ad_\varphi$, whence we get $D = ad_\psi + ad_\phi = ad_{\psi+\phi}$. Thus D is also inner.

Now suppose T is a derivation from $\mathfrak{A}$ to E^*. Then $a \mapsto T(a)|_{E_1}$ is a derivation into E_1^*, which is inner by the first part of the proof. Moreover, since $\mathfrak{A} \cdot (E/E_1) = \{0\}$, we can show similarly to above that T is inner. So $H^1(\mathfrak{A}, E^*) = 0$.

Hence for every Banach $\mathfrak{A}$ bimodule E we have shown that $H^1(\mathfrak{A}, E^*) = 0$. Therefore, $\mathfrak{A}$ is amenable. $\square$

Suppose $\mathfrak{A}$ is a Banach algebra with a bounded approximate identity and $\mathfrak{A}$ is contained as a closed ideal in some other algebra $\mathfrak{B}$. Then, if E is a pseudo-unital Banach $\mathfrak{A}$-bimodule, we can make it into a Banach $\mathfrak{B}$-bimodule as follows: consider $b \in \mathfrak{B}$ and $x \in E$. By Cohen's Factorization Theorem (Theorem 2.20) we can find $a \in \mathfrak{A}$ and $y \in E$ such that $x = a \cdot y$. Since $\mathfrak{A}$ is an ideal of $\mathfrak{B}$ and hence $ba \in \mathfrak{A}$ we can define $b \cdot x$ by setting it equal to $(ba) \cdot y$. We need to show that this is well-defined. Let $(e_\alpha)_\alpha$ be a bounded approximate identity for $\mathfrak{A}$. Suppose $x = a \cdot y = a' \cdot y'$ for $a, a' \in \mathfrak{A}$ and $y, y' \in E$. Then we have $(ba) \cdot y = \lim_\alpha be_\alpha a \cdot y = \lim_\alpha be_\alpha(a' \cdot y') = ba' \cdot y'$. So it follows that the definition of $b \cdot x$ is independent of the factorization of x. It is easy to check that the above definition of $b \cdot x$ satisfies all the required properties of a module action, and that $\|b \cdot x\| \leq \|b\|\|x\| \sup_\alpha \|e_\alpha\|$; hence E is a left Banach $\mathfrak{B}$-module. Similarly we can make E into a right Banach $\mathfrak{B}$-module.

This construction leads us to question whether a derivation can be extended from a subalgebra to the algebra containing it. The next theorem describes a situation in which such an extension exists and is unique.

If $\mathfrak{A}, \mathfrak{B}$ are two Banach algebras such that $\mathfrak{A}$ is a closed ideal of $\mathfrak{B}$, we define the **strict topology** on $\mathfrak{B}$ with respect to $\mathfrak{A}$ to be the weakest topology such that for each $a \in \mathfrak{A}$ the maps $b \mapsto ab$ and $b \mapsto ba$ (where $b \in \mathfrak{B}$) are both continuous. It is clear that this topology is generally weaker than the norm topology. Note that if $b_\lambda \to b$ in the strict topology on $\mathfrak{B}$ and E is a pseudo-unital Banach $\mathfrak{A}$-bimodule, then $b_\lambda \cdot v \to b \cdot v$ for every $v \in E$. This follows from the action of $\mathfrak{B}$ on E defined earlier; if we write $v = a \cdot u$ for $a \in \mathfrak{A}$ and $u \in E$ then $b_\lambda \cdot v = (b_\lambda a) \cdot u \to ba \cdot u$ (by the definition of the strict topology and the fact that the action of $\mathfrak{A}$ on E is continuous). Similarly, $v \cdot b_\lambda \to v \cdot b$.

Theorem 2.25. *Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity. Suppose $\mathfrak{A}$ is contained as a closed ideal in a Banach algebra $\mathfrak{B}$. Let E be a pseudo-unital Banach $\mathfrak{A}$-bimodule, and let $D \in Z^1(\mathfrak{A}, E^*)$. Then E is a Banach $\mathfrak{B}$-bimodule, and there is a unique $T \in Z^1(\mathfrak{B}, E^*)$ such that $T|_{\mathfrak{A}} = D$ and T is continuous with respect to the strict topology on $\mathfrak{B}$ and the weak*-topology on E^*.*

Proof. Let $(e_\alpha)_\alpha$ be a bounded approximate identity for $\mathfrak{A}$. The construction which makes E into a Banach $\mathfrak{B}$-bimodule was described in the comments leading up to this theorem. If T is a derivation on $\mathfrak{B}$ note that in particular it must satisfy $T(be_\alpha) = T(b)\cdot e_\alpha + b\cdot T(e_\alpha)$ for any $b \in \mathfrak{B}$. If T also extends D, this is equivalent to $D(be_\alpha) = T(b) \cdot e_\alpha + b \cdot D(e_\alpha)$ ($be_\alpha \in \mathfrak{A}$ since $\mathfrak{A}$ is an ideal). Recall that $T(b) \cdot e_\alpha \to T(b)$ in the weak* topology on E^*. This suggests that, if $\{[D(be_\alpha) - b \cdot D(e_\alpha)]\}_\alpha$ has a weak*-limit ϕ_b, then we must have $T(b) = \phi_b$. Thus, since $be_\alpha, e_\alpha \in \mathfrak{A}$ for each α and $T|_{\mathfrak{A}} = D$, T is determined by its values on $\mathfrak{A}$; hence, T is unique.

Fix $b \in \mathfrak{B}$ and consider the net $(D(be_\alpha) - b \cdot D(e_\alpha))_\alpha$. We want to show that this net has a weak* limit. Let $u \in E$. By Cohen's Factorization Theorem (Theorem 2.20) we can find $a \in \mathfrak{A}$ and $v \in E$ such that $u = v \cdot a$. Note that

$$\begin{aligned}
[D(be_\alpha) - b \cdot D(e_\alpha)](v \cdot a) &= [a \cdot D(be_\alpha)](v) - [a \cdot b \cdot D(e_\alpha)](v) \\
&\qquad \text{(by the definition of the action of } \mathfrak{A} \text{ on } E^*) \\
&= [D(abe_\alpha) - D(a)be_\alpha](v) - [D(abe_\alpha) - D(ab)e_\alpha](v) \\
&\qquad \text{(since } D \text{ is a derivation)} \\
&= [D(ab)e_\alpha](v) - [D(a)be_\alpha](v) \\
&= [D(ab)e_\alpha](v) - [D(a)](be_\alpha \cdot v)
\end{aligned}$$

But $D(ab)e_\alpha \overset{wk*}{\to} D(ab)$ and $e_\alpha \cdot v \to v$ (since $(e_\alpha)_\alpha$ is a bounded approximate identity for $\mathfrak{A}$ and E is a pseudo-unital module). Therefore,

$$[D(be_\alpha) - b \cdot D(e_\alpha)](a \cdot v) \to [D(ab)](v) - [D(a)](b \cdot v).$$

Define $T : \mathfrak{B} \to E^*$ by $b \mapsto \text{wk*-}\lim_\alpha [D(be_\alpha) - b \cdot D(e_\alpha)]$. We will show that this map satisfies all the requirements of the theorem.

Consider $a \in \mathfrak{A}$. Then

$$\begin{aligned} T(a) &= \text{wk*-}\lim_{\alpha}[D(ae_\alpha) - a \cdot D(e_\alpha)] && \text{(by definition)} \\ &= \text{wk*-}\lim_{\alpha}[a \cdot D(e_\alpha) + D(a) \cdot e_\alpha - a \cdot D(e_\alpha)] && \text{(since } D \text{ is a derivation)} \\ &= \text{wk*-}\lim_{\alpha}[D(a) \cdot e_\alpha] \\ &= D(a). \end{aligned}$$

Hence $T|_{\mathfrak{A}} = D$.

Next we show that T is continuous with respect to the strict topology on $\mathfrak{B}$ and the weak* topology on E^*. Let $(b_\lambda)_\lambda$ be a net which converges to some $b \in \mathfrak{B}$ in the strict topology. Consider $u \in E$ and suppose $u = a_1 \cdot v \cdot a_2$ for $a_1, a_2 \in \mathfrak{A}$, and $v \in E$ (we can write u this way since E is pseudo-unital). Then from before $[T(b_\lambda)](u) = [T(b_\lambda)](a_1 \cdot v \cdot a_2) = [D(a_2 b_\lambda)](a_1 v) - [D(a_1)](b_\lambda a_1 v)$. But $a_2 b_\lambda \to a_2 b$ and $b_\lambda a_1 \to b a_1$ (by the definition of the strict topology), so it follows that $[T(b_\lambda)](u) \to [T(b)](u)$. Therefore, $T(b_\lambda) \to T(b)$ in the weak* topology whenever $b_\lambda \to b$ in the strict topology on $\mathfrak{B}$.

Finally, we need to check that T is a derivation. Let $b_1, b_2 \in \mathfrak{B}$. Since $(e_\alpha)_\alpha$ is a bounded approximate identity for $\mathfrak{A}$, we have that $be_\alpha \to b$ in the strict topology for any $b \in \mathfrak{B}$. So we can write

$$\begin{aligned} T(b_1 b_2) &= \text{wk*-}\lim_{\alpha} \text{wk*-}\lim_{\beta} T((b_1 e_\alpha)(b_2 e_\beta)) \\ &\qquad \text{(by continuity of } T \text{ with respect to the strict topology)} \\ &= \text{wk*-}\lim_{\alpha} \text{wk*-}\lim_{\beta} D((b_1 e_\alpha)(b_2 e_\beta)) \\ &\qquad \text{(since } \mathfrak{A} \text{ is an ideal of } \mathfrak{B}\text{, and } T|_{\mathfrak{A}} = D\text{)} \\ &= \text{wk*-}\lim_{\alpha} \text{wk*-}\lim_{\beta} b_1 e_\alpha D(b_2 e_\beta) + D(b_1 e_\alpha) b_2 e_\beta \\ &\qquad \text{(since } D \text{ is a derivation)} \\ &= b_1 T(b_2) + T(b_1) b_2 \\ &\qquad \text{(by continuity with respect to the strict topology)} \end{aligned}$$

Therefore, T is a derivation. □

In particular, if G is a locally compact group, we can apply the above theorem for $\mathfrak{A} = L^1(G)$, $\mathfrak{B} = M(G)$, where $M(G)$ is the set of all complex, regular Borel measures on G. Moreover, the extension T mentioned in the theorem above is uniquely determined by its values on $\{\delta_g : g \in G\}$, since such measures are weak*-dense in $M(G)$.

We are finally able to connect the definition of amenability from groups to Banach algebras.

Theorem 2.26. *[Johnson] Let G be a locally compact group. Then G is amenable if and only if $L^1(G)$ is amenable.*

Proof. Suppose G is amenable. Let E be a pseudo-unital Banach $L^1(G)$ bimodule. We want to show that $H^1(L^1(G), E^*) = \{0\}$. Suppose D is a derivation from $L^1(G)$ to E^*; as described by Theorem 2.25 above we can extend D to a derivation T from $M(G)$ to E^*.

Define an action of G on E^* by $g \cdot \phi = [\delta_g \cdot \phi + T(\delta_g)] \cdot \delta_{g^{-1}}$. In order to conclude that this does indeed define an action, the only thing which is not obvious is that $g \cdot (h \cdot \phi) = (gh) \cdot \phi$ for all $g, h \in G$ and $\phi \in E^*$.

$$\begin{aligned}
(gh) \cdot \phi &= [\delta_{gh} \cdot \phi + T(\delta_{gh})] \cdot \delta_{(gh)^{-1}} \\
&= [\delta_{gh} \cdot \phi + \delta_g T(\delta_h) + T(\delta_g)\delta_h] \cdot \delta_{(gh)^{-1}} \\
&\qquad \text{(since } T \text{ is a derivation and } \delta_{gh} = \delta_g * \delta_h\text{)} \\
&= \delta_g \cdot \delta_h \cdot \phi \cdot \delta_{h^{-1}}\delta_{g^{-1}} + \delta_g T(\delta_h) \cdot \delta_{h^{-1}}\delta_{g^{-1}} + T(\delta_g) \cdot \delta_{g^{-1}} \\
&= \delta_g \cdot [\delta_h \cdot \phi \cdot \delta_{h^{-1}} + T(\delta_h) \cdot \delta_{h^{-1}}] \cdot \delta_{g^{-1}} + T(\delta_g) \cdot \delta_{g^{-1}} \\
&= [\delta_g \cdot (h \cdot \phi) + T(\delta_g)] \cdot \delta_{g^{-1}} \\
&= g \cdot (h \cdot \phi)
\end{aligned}$$

Also, the action of G on E^* defined above is affine, since for $g \in G$ and $\phi_1, \phi_2 \in E^*$ and $t \in [0, 1]$ we have

$$\begin{aligned}
t[g \cdot \phi_1] + (1-t)[g \cdot \phi_2] &= t[\delta_g \cdot \phi_1 + T(\delta_g)] \cdot \delta_{g^{-1}} + (1-t)[\delta_g \cdot \phi_2 + T(\delta_g)] \cdot \delta_{g^{-1}} \\
&= \delta_g \cdot [t\phi_1 + (1-t)\phi_2] \cdot \delta_{g^{-1}} + (t + (1-t))T(\delta_g) \cdot \delta_{g^{-1}} \\
&= g \cdot [t\phi_1 + (1-t)\phi_2]
\end{aligned}$$

Let K be the weak*-closed convex hull of $\{T(\delta_g) \cdot \delta_{g^{-1}} : g \in G\}$. Then K is weak* compact (since bounded in norm) and convex by definition. We want to use Day's Fixed Point Theorem, so we need $g \cdot \phi \in K$ for $g \in G$ and $\psi \in K$. To prove this, it is enough to show that $g \cdot (T(\delta_h) \cdot \delta_{h^{-1}})$ is in K for any $g, h \in G$ (the result for any $\psi \in K$ follows by continuity and linearity). We have

$$\begin{aligned}
g \cdot (T(\delta_h) \cdot \delta_{h^{-1}}) &= \delta_g \cdot T(\delta_h) \cdot \delta_{h^{-1}} \cdot \delta_{g^{-1}} + T(\delta_g) \cdot \delta_{h^{-1}} \cdot \delta_{g^{-1}} \\
&= T(\delta_g \delta_h) \cdot \delta_{(gh)^{-1}} - T(\delta_g) \cdot \delta_h \cdot \delta_{h^{-1}}\delta_{g^{-1}} + T(\delta_g) \cdot \delta_{g^{-1}} \\
&= T(\delta_{gh}) \cdot \delta_{(gh)^{-1}}
\end{aligned}$$

Hence $g \cdot (T(\delta_h) \cdot \delta_{h^{-1}}) = T(\delta_{gh}) \cdot \delta_{(gh)^{-1}} \in K$.

We also need to check that $(g, k) \mapsto g \cdot k$ is separately continuous. Fix $\psi_0 \in K$ and suppose $(g_\alpha)_\alpha$ is a net converging to $g \in G$. Then $g_\alpha \cdot \psi_0 = [\delta_{g_\alpha} \cdot \psi_0 + T(\delta_{g_\alpha})] \cdot \delta_{(g_\alpha)^{-1}}$. Since $\delta_{g_\alpha} \to \delta_g$ in the strict topology on $M(G)$ and T is continuous with respect to the strict topology as shown in Theorem 2.25, we know that $T(\delta_{g_\alpha}) \overset{wk*}{\to} T(\delta_g)$. From the comment made before Theorem 2.25 we also know that $\delta_{g_\alpha} \cdot v \to \delta_g \cdot v$ for every $v \in E$, whence it follows that $\delta_{g_\alpha} \cdot \psi_0 \overset{wk*}{\to} \delta_g \cdot \psi_0$. Let $\phi_\alpha = \delta_{g_\alpha} \cdot \psi_0 + T(\delta_{g_\alpha})$ and $\phi = \delta_g \cdot \psi_0 + T(\delta_g)$. We have shown above that $\phi_\alpha \to \phi$ in the weak* topology; we still need to show that $\phi_\alpha \cdot \delta_{(g_\alpha)^{-1}} \overset{wk*}{\to} \phi \cdot \delta_{g^{-1}}$. Fix $v \in E$. Then $[\phi_\alpha \cdot \delta_{(g_\alpha)^{-1}}](v) = \phi_\alpha(\delta_{(g_\alpha)^{-1}} \cdot v)$. We can write $\|(\phi_\alpha(\delta_{(g_\alpha)^{-1}} \cdot v) - \phi(\delta_{g^{-1}} \cdot v)\| \leq \|\phi_\alpha\| \|\delta_{(g_\alpha)^{-1}} \cdot v - \delta_{g^{-1}} \cdot v\| + \|\phi_\alpha(\delta_{g^{-1}} \cdot v) - \phi(\delta_{g^{-1}} \cdot v)\|$. But $\delta_{(g_\alpha)^{-1}} \cdot v \to \delta_{g^{-1}} \cdot v$ and $(\phi_\alpha)_\alpha$ is bounded, so this allows us to conclude that $\phi_\alpha \cdot \delta_{(g_\alpha)^{-1}} \overset{wk*}{\to} \phi \cdot \delta_{g^{-1}}$. This concludes the proof of the fact that $g_\alpha \cdot \psi_0 \to g \cdot \psi_0$.

Fix $g_0 \in G$ and suppose $(\psi_\beta)_\beta$ is a net converging to $\psi \in K$ in the weak* topology. Since the action of $L^1(G)$ on E is continuous in the norm topology, the definition of the action of $L^1(G)$ on E^* implies $\delta_g \cdot \psi_\beta \overset{wk*}{\to} \delta_g \cdot \psi$. From this it follows immediately that $g_0 \cdot \psi_\beta \overset{wk*}{\to} g_0 \cdot \psi$, as desired.

Therefore, all the requirements of Day's Fixed Point Theorem are satisfied. Hence there exists a $\phi_0 \in K$ such that $g \cdot \phi_0 = \phi_0$ for all $g \in G$. Using the definition of the action we get that $\delta_g \cdot \phi_0 + T(\delta_g)] \cdot \delta_{g^{-1}} = \phi_0$, hence $T(\delta_g) = \phi_0 \cdot \delta_g - \delta_g \cdot \phi_0$ for all $g \in G$. Recall that the set $\{\delta_g : g \in G\}$ is weak* dense in $M(G)$ and T is continuous with respect to the weak* topology on E^*; it follows that $T(\mu) = \phi_0 \cdot \mu - \mu \cdot \phi_0$ for any $\mu \in M(G)$. Finally, since $D = T|_{L^1(G)}$, it follows that D is inner. Therefore, all the continuous derivations on $L^1(G)$ are inner, and hence $L^1(G)$ is amenable.

Conversely, suppose $L^1(G)$ is amenable. Define an $L^1(G)$-bimodule action on $L^\infty(G)$ by $\phi \cdot \alpha = \phi * \alpha$ and $\alpha \cdot \phi = (\int_G \phi(g)\, dg)\alpha$ for $\phi \in L^1(G)$ and $\alpha \in L^\infty(G)$. By the Hahn-Banach theorem, we can find some $n_0 \in L^\infty(G)^*$ such that $n_0(\mathbf{1}) = 1$. Define $D : L^1(G) \to L^\infty(G)^*$ by $\phi \mapsto \phi \cdot n_0 - n_0 \cdot \phi$.

Let $E = L^\infty(G)/\mathbb{C}\mathbf{1}$. Then E is a quotient module of $L^\infty(G)$. Moreover, note that for $\phi \in L^1(G)$ we have

$$\begin{aligned}[D(\phi)](\mathbf{1}) &= [\phi \cdot n_0](\mathbf{1}) - [n_0 \cdot \phi](\mathbf{1}) \\ &= n_0(\mathbf{1} \cdot \phi) - n_0(\phi \cdot \mathbf{1}) \\ &= n_0((\textstyle\int_G \phi(g)\, dg)\mathbf{1}) - n_0(\phi * \mathbf{1})\end{aligned}$$

Since $\phi * \mathbf{1} = \int_G \phi(g)\, dg$, it follows that $[D(\alpha)](\mathbf{1}) = 0$, and hence $D|_{\mathbb{C}\mathbf{1}} = 0$. Therefore, we can consider D as a derivation from $L^1(G)$ to E^*. But since $L^1(G)$ is amenable, every derivation into a dual group is inner; hence there exists an $r_0 \in E^*$ such that $D(\phi) = \phi \cdot r_0 - r_0 \cdot \phi$ for every $\phi \in L^1(G)$. Comparing with the previous definition of D, we get $\phi \cdot n_0 - n_0 \cdot \phi = \phi \cdot r_0 - r_0 \cdot \phi$ for any $\phi \in L^1(G)$. Hence $\phi \cdot (n_0 - r_0) = (n_0 - r_0) \cdot \phi$, which implies $(n_0 - r_0)(\alpha \cdot \phi) = (n_0 - r_0)(\phi \cdot \alpha)$ for any $\alpha \in L^\infty(G)$ and $\phi \in L^1(G)$.

Let $n = n_0 - r_0$. From the above observation $n(\alpha \cdot \phi) = n(\phi \cdot \alpha)$ for any $\alpha \in L^\infty(G)$ and $\phi \in L^1(G)$. In particular, this is true for $\varphi \in P(G)$. Since $\alpha \cdot \varphi = \|\varphi\|_1 \alpha = \alpha$ and $\varphi \cdot \alpha = \varphi * \alpha$ (by definition), it follows that $n(\varphi * \alpha) = n(\alpha)$ for any $\varphi \in P(G)$ and $\alpha \in L^\infty(G)$. Thus, by an argument similar to the proof for Theorem 2.9, we can show that n is left invariant. Note however that n is not necessarily a mean (since it might not be a positive functional).

Finally, we use the n obtained above to define a left invariant mean on $L^\infty(G)$. Since $L^\infty(G)$ is an abelian $\mathcal{C}^*$-algebra and $\Sigma_{L^\infty(G)}$ is compact, the Gelfand transform is an isometric *-isomorphism from $L^\infty(G)$ to $\mathcal{C}(\Sigma_{L^\infty(G)})$. It follows that any linear functional on $L^\infty(G)$ can be identified with a measure on $\Sigma_{L^\infty(G)}$ by the Riesz Representation Theorem. Then $|n|$ is a positive linear functional on $L^\infty(G)$, and since n is left invariant so is $|n|$. Recall that a mean is a positive linear functional which evaluates to 1 at $\mathbf{1}$; so all we need to do to obtain a mean from $|n|$ is to scale it. Let $m = (|n|(\mathbf{1}))^{-1}|n|$. Then m is a left invariant mean on $L^\infty(G)$, and therefore G is amenable. $\square$

Chapter 3

Operator Algebras and Invariant Subspaces

We are now ready to apply the concepts of the previous chapter to operator algebras. Let $\mathcal{H}$ be a Hilbert space, and denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. Recall that $\mathcal{B}(\mathcal{H})$ equipped with the usual operator norm and the involution given by the adjoint operation is a $\mathcal{C}^*$-algebra. We will most often be working with subalgebras of $\mathcal{B}(\mathcal{H})$ which are not necessarily self-adjoint.

In this chapter we also discuss the invariant subspaces of an operator algebra. We will find that it is useful to be able to describe the invariant subspaces of an algebra made up of specific types of operators, as well as to recognize that from a description of the set of invariant subspaces of an algebra we can occasionally draw conclusions about the algebra itself.

Definition 3.1. *For $T \in \mathcal{B}(\mathcal{H})$ we say that a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is an* ***invariant subspace*** *if $\mathcal{M}$ is closed and $T\mathcal{M} \subseteq \mathcal{M}$.*
For $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ we define $Lat\mathfrak{A} = \{\mathcal{M} : \mathcal{M}$ is an invariant subspace for all $T \in \mathfrak{A}\}$.

Note that if we order the subspaces by inclusion and define $\mathcal{M} \wedge \mathcal{N} = \mathcal{M} \cap \mathcal{N}$ and $\mathcal{M} \vee \mathcal{N} = \overline{\mathcal{M} + \mathcal{N}}$, then the above set does indeed form a lattice. Moreover, the lattice is non-empty since $\{0\}$ and $\mathcal{H}$ are invariant for any $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$.

A chain of subspaces in $\operatorname{Lat}\mathfrak{A}$ is complete if it is closed under arbitrary intersections and closed linear spans. A complete chain $\mathcal{C}$ which contains $\{0\}$ and $\mathcal{H}$ is called a nest.

For every $\mathcal{N}$ in a chain, we define the predecessor of $\mathcal{N}$ to be $\mathcal{N}_{-} = \vee\{\mathcal{M} \in \mathcal{C} : \mathcal{M} \subsetneq \mathcal{N}\}$ If for every $\mathcal{N} \in \mathcal{C}$ we have that $\mathcal{N} = \mathcal{N}_{-}$, then $\mathcal{C}$ is called a **continuous nest**. If $\mathcal{N} \neq \mathcal{N}_{-}$, then $\mathcal{N} \ominus \mathcal{N}_{-}$ is an atom of the chain. A chain is maximal in $\operatorname{Lat}\mathfrak{A}$ if it is not contained in any other chain in $\operatorname{Lat}\mathfrak{A}$. A chain is maximal in the family of all chains if and only if it is complete and all its atoms are one-dimensional ([18], Theorem 5.10).

We can make $\mathcal{H}$ into a module for $\mathfrak{A}$ by defining $T \cdot u = T(u)$ for all $T \in \mathfrak{A}$ and $u \in \mathcal{H}$. It is then easy to see that the submodules of $\mathcal{H}$ are exactly the spaces in $\operatorname{Lat}\mathfrak{A}$. A module $\mathcal{M}$ is cyclic (with cyclic vector v) if $\mathcal{M} = \overline{\mathfrak{A}v}$, and it is irreducible if every $v \in \mathcal{M}$ is cyclic for $\mathcal{M}$. Irreducible modules will prove to be particularly important when we examine algebras of compact operators.

If $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces of $\mathcal{H}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{H} = \mathcal{M} + \mathcal{N}$ then we will write $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ (even though $\mathcal{M}$ and $\mathcal{N}$ might not be orthogonal). In particular, if $\mathcal{M} \in \operatorname{Lat}\mathfrak{A}$ then we say that $\mathcal{M}$ is complemented if there exists a subspace $\mathcal{N} \in \operatorname{Lat}\mathfrak{A}$ such that $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$.

There is a connection between complemented modules and idempotents in $\mathcal{B}(\mathcal{H})$. First recall that $P \in \mathcal{B}(\mathcal{H})$ is an idempotent if $P^2 = P$. In general, in a Hilbert space the term projection is reserved for self-adjoint idempotents. However, in keeping with Gifford ([8]), we will refer to idempotents as projections and we will use the term “orthogonal projection” for self-adjoint idempotents.

If $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces then there is a projection P which has range $\mathcal{M}$ and kernel $\mathcal{N}$, called the projection on $\mathcal{M}$ along $\mathcal{N}$ (see [17]).

On the other hand, suppose P is a projection onto a submodule $\mathcal{M}$, and let $\mathcal{N} = \ker P$. Then clearly $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. Moreover, $\mathcal{N}$ is a submodule if and only if $P \in \mathfrak{A}'$ (as we shall prove below, following [2]).

First suppose that $\mathcal{N}$ is an invariant subspace of $\mathfrak{A}$. Consider $v \in \mathcal{H}$. Then $Pv \in \mathcal{M}$ and $(I-P)v \in \mathcal{N}$ (follows from the definition of the subspaces). Fix $T \in \mathfrak{A}$. Since the subspaces are invariant we have that $TPv \in \mathcal{M}$ and $T(I-P)v \in \mathcal{N}$. But note that $Tv = TPv + T(I-P)v$. Since TPv is in the range of P we have $PTPv = TPv$; and $T(I-P)v$ is in the kernel of P, so $PT(I-P)v = 0$. Hence, we can apply P to both sides of the equality to obtain $PTv = TPv$. Since $v \in \mathcal{H}$ was arbitrary, we conclue $PT = TP$, and hence $P \in \mathfrak{A}'$ as desired.

On the other hand, if $P \in \mathfrak{A}'$ then for any $u \in \mathcal{N}$ we have $Pu = 0$ (by definition of the subspace), and so for any $T \in \mathfrak{A}$ we have $PTu = TPu = 0$, and hence $Tu \in \ker P$. Therefore $\mathcal{N} = \ker P$ is invariant for $\mathfrak{A}$.

Finally, note that there can be multiple projections onto a module $\mathcal{M}$. If $\mathcal{N}$ is a complementary module of $\mathcal{M}$, the matrix of a projection onto $\mathcal{M}$ has the form $\begin{array}{cc} \begin{bmatrix} I & E \\ 0 & 0 \end{bmatrix} & \begin{array}{c} \mathcal{M} \\ \mathcal{N} \end{array} \end{array}$, where $E = 0$ if and only if the projection is along $\mathcal{N}$. In particular, note that if $\mathcal{M}$ and $\mathcal{N}$ are not orthogonal, the projection on $\mathcal{M}$ along $\mathcal{N}$ is not self-adjoint.

We have already observed that the projections we deal with might not be self-adjoint. However, the lemma below tells us that, under certain conditions, we can find a similarity transform which orthogonalizes all the projections in a given set. Even though the conditions on the set of idempotents might seem restrictive, this turns out to be a very useful result. If P, Q are two operators, we define the symmetric difference to be $P \triangle Q = P + Q - 2PQ$.

Lemma 3.2. *Let $\mathcal{P} \subseteq \mathcal{B}(\mathcal{H})$ be a uniformly bounded set of commuting idempotents, closed under symmetric differences. Then there exists a similarity $S \in \mathcal{B}(\mathcal{H})$ such that SPS^{-1} is self-adjoint for all $P \in \mathcal{P}$. In particular, if $\|P\| \leq K$ for all $P \in \mathcal{P}$ then S can be chosen with $\|S\|\|S^{-1}\| \leq (1 + 2K)^2$.*

Proof. Let $G = \{I - 2P : P \in \mathcal{P}\}$. We claim that G is a group under multiplication. Pick any $P \in \mathcal{P}$. Since $\mathcal{P}$ is closed under symmetric difference, we get that $0 = P + P - 2P^2 = P \triangle P$ (since P is idempotent, $P^2 = P$) is in $\mathcal{P}$, and hence $I \in G$. Also, since P is idempotent, $(I - 2P)^2 = I$ and hence $I - 2P$ is invertible. Finally, for any $Q \in \mathcal{P}$ we have that $(I - 2P)(I - 2Q) = I - 2(P + Q - 2PQ) \in G$ since $\mathcal{P}$ is abelian and closed under symmetric differences. Therefore G is a group under multiplication. Moreover, since we are given that the idempotents commute, G is in fact abelian.

Consider G with the discrete topology, and a representation of G onto $\mathcal{B}(\mathcal{H})$ given by the identity map. Since any locally compact abelian group is amenable, we can use Theorem 2.16 to find a similarity S such that $S(I - 2P)S^{-1} = I - 2SPS^{-1}$ is unitary for each $P \in \mathcal{P}$. Hence $[I - 2SPS^{-1}]^{-1} = [I - 2(SPS^{-1})]^*$. On the other hand, $[I - 2SPS^{-1}]^2 = I - 4SPS^{-1} + 4SPS^{-1}SPS^{-1} = I$, so $[I - 2SPS^{-1}]^{-1} = I - 2SPS^{-1}$

as well. Therefore $I - 2SPS^{-1}$ is self-adjoint, whence it follows that SPS^{-1} is also self-adjoint. Moreover, since $\|1 - 2P\| \leq 1 + 2K$ for any $P \in \mathcal{P}$, we have that $\|S\|\|S^{-1}\| \leq (1 + 2K)^2$. □

3.1 Types of Operators

In this section we discuss properties of compact, triangular and quasitriangular operators. This will lay the foundation for the proofs presented in the next chapter, when we examine the conditions required for specific types of operator algebras to be similar to $\mathcal{C}^*$-algebras.

Definition 3.3. *An operator K is **compact** if $\overline{Kb_1(\mathcal{H})}$ is compact (where $b_1(\mathcal{H})$ is the unit ball of $\mathcal{H}$).*

The set of compact operators is denoted by $\mathcal{K}(\mathcal{H})$. It is a standard result that any compact operator on a Hilbert space can be written as a limit of finite rank operators. The set of compact operators is an ideal in $\mathcal{B}(\mathcal{H})$.

Every compact operator has a non-trivial invariant subspace. In fact, $\operatorname{Lat} T$ contains a maximal subspace chain (see [18], p. 89). It is, however, Lomonosov's Lemma which will prove to be key for the discussion of operator algebras contained in $\mathcal{K}(\mathcal{H})$. In order to avoid introducing too many new concepts we state a few theorems without proof and concentrate on the parts of Lomonosov's Lemma that we will need later.

Theorem 3.4. *[[18], Corollary 2.13] Let $A \in \mathcal{B}(\mathcal{H})$. If f is analytic on $\sigma(A)$ and the bounded components of $\rho(A)$ then $\operatorname{Lat} A \subset \operatorname{Lat} f(A)$.*

We define $\mathcal{H}^{(n)}$ to be the direct sum of n copies of $\mathcal{H}$. To extend this to the countable case use the l^2 direct sum; that is,

$$\mathcal{H}^{(\infty)} = \sum_{i\in\mathbb{N}}^{\oplus_2} \mathcal{H} = \{(v_1, v_2, \ldots) : v_i \in \mathcal{H} \text{ and } \sum_{i=1}^{\infty} \|v_i\|^2 < \infty\}$$

If T is an operator in $\mathcal{B}(\mathcal{H})$, then we can obtain a corresponding operator $T^{(n)}$ in $\mathcal{H}^{(n)}$ by applying T to each component. Finally, if $\mathfrak{A}$ is an algebra of operators, then we define $\mathfrak{A}^{(n)}$ to be the algebra $\{T^{(n)} : T \in \mathfrak{A}\}$.

Theorem 3.5. *[[18], Corollary 7.2] Let $\mathfrak{A}$ be an algebra of operators containing the identity. The weak operator closure of $\mathfrak{A}$ is given by*

$$\{T \in \mathcal{B}(\mathcal{H}) : \operatorname{Lat}\mathfrak{A}^{(n)} \subset \operatorname{Lat} T^{(n)} \text{ for all } n \in \mathbb{N}\}.$$

Combining the two previous theorems, if K is a compact operator, and $\mathfrak{A}$ is the algebra generated by I and K, then whenever f is a function which is analytic on $\sigma(K)$ we have that $f(K)$ is in the weak closure of $\mathfrak{A}$ (since for each n we have $K^{(n)}$ is compact, so $\rho(K^{(n)})$ has no unbounded components; Theorem 3.4 gives us that $\mathrm{Lat}\, K^{(n)} \subset \mathrm{Lat}\, f(K)^{(n)}$, and Theorem 3.5 gives us that $f(K) \in \overline{\mathfrak{A}}^{WOT}$). An alternate way of establishing this result is to recall that $f(K) \in \overline{\mathfrak{A}}^{\|\cdot\|}$ by the holomorphic functional calculus.

Theorem 3.6. *[[18], Theorem 8.12] Let $\mathfrak{A}$ be an algebra such that $I \in \mathfrak{A}$ and $Lat\mathfrak{A} = \{\{0\}, \mathcal{H}\}$. If $\mathfrak{A}$ contains a finite rank operator, then $\mathfrak{A}$ is weakly dense in $\mathcal{B}(\mathcal{H})$.*

The above theorem and the comment preceding it are the tools we need to prove Lomonosov's Lemma.

Theorem 3.7. *[Lomonosov's Lemma] Suppose an algebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ contains the identity operator and satisfies $Lat\mathfrak{A} = \{\{0\}, \mathcal{H}\}$. If $\mathfrak{A}$ also contains a non-zero compact operator, then $\mathfrak{A}$ is weakly dense in $\mathcal{B}(\mathcal{H})$.*

Proof. This proof is from [18], Lemma 8.22. Let K be a non-zero compact operator in $\mathfrak{A}$. Suppose that we can show that there is an $A \in \mathfrak{A}$ such that 1 is an eigenvalue of AK. Since AK is compact, 1 is an isolated point of the spectrum. By the Riesz Decomposition Theorem, we can find a function f which is holomorphic on an open set containing the spectrum such that $P = f(AK)$ is a projection and $\sigma(AK|_{P\mathcal{H}}) = \{1\}$. But $AK|_{P\mathcal{H}}$ is a compact operator, so since its spectrum does not contain 0 it must have finite rank. Moreover, $f(AK)$ is in the weak closure of $\mathfrak{A}$ by the comment following Theorem 3.5. Then by Theorem 3.6 we get that the weak closure of $\mathfrak{A}$ must be all of $\mathcal{B}(\mathcal{H})$.

So all that is left is to construct an $A \in \mathfrak{A}$ such that AK has 1 as an eigenvalue, i.e. $AKv_1 = v_1$ for some $v_1 \in \mathcal{H}$. Suppose that ϕ is a continuous function on $\mathcal{H}$ given by $\phi(v) = A_v Kv$ for some $A_v \in \mathfrak{A}$. If moreover there exists a compact, convex subset $\mathcal{C}$ of $\mathcal{H}$ such that $\phi(\mathcal{C}) \subset \mathcal{C}$, then the Schauder Fixed Point Theorem gives us that ϕ has a fixed point in $\mathcal{C}$, and the desired result follows. So the goal is to define a suitable function ϕ.

By scaling if necessary, we can assume without loss of generality that $\|K\| = 1$. Pick a $v_0 \in \mathcal{H}$ such that $\|v_0\| > 1$ and $\|Kv_0\| > 1$. Let $\mathcal{S}$ be the closed ball of radius 1 centered at v_0. So $\overline{K\mathcal{S}}$ is compact (since K is a compact operator). Moreover, for any $v \in \mathcal{S}$ we have $|\|Kv\| - \|Kv_0\|| \leq \|Kv - Kv_0\| \leq \|v - v_0\| \leq 1$ (recall $\|K\| = 1$), and so since $\|Kv_0\| > 1$ we get $\|Kv\| > 0$; hence $0 \notin \overline{K\mathcal{S}}$. But then $\overline{K\mathcal{S}} \subset \mathcal{H}\backslash\{0\} \subset \bigcup_{A \in \mathfrak{A}} \{u \in \mathcal{H} : \|Au - v_0\| < 1\}$.

Note that the latter inclusion follows because $\operatorname{Lat}\mathfrak{A} = \{\{0\}, \mathcal{H}\}$, so for every $u \in \mathcal{H} \setminus \{0\}$ we have $\overline{\mathfrak{A}u} = \mathcal{H}$, and in particular v_0 can be written as a limit of elements in $\mathfrak{A}u$. Since any open cover of $\overline{K\mathcal{S}}$ has a finite subcover, we can find $A_1, A_2, \ldots A_n$ such that $\overline{K\mathcal{S}} \subset \bigcup\limits_{i=1}^{n} \{u \in \mathcal{H} : \|A_i u - v_0\| < 1\}$.

For $i = 1, \ldots, n$ define $\alpha_i(w) = \max\{0, 1 - \|A_i w - v_0\|\}$ for $w \in \overline{K\mathcal{S}}$. Note from the definition that $0 \leq \alpha_i(w) \leq 1$. Moreover, for any w we have that $w \in A_j$ for some j, whence $\|A_j w - v_0\| < 1$ so $\alpha_j(w) \neq 0$. Thus we can scale the α_i's such that they add up to 1; define $\beta_i(w) = \alpha_i(w)/(\sum\limits_{k=1}^{n} \alpha_k(w))$.

Finally, for $u \in \mathcal{S}$ define $\psi(u) = \sum\limits_{i=1}^{n} \beta_i(Ku) A_i K u$. Since the β_i's are defined to be between 0 and 1 and to add up to 1, the range of ψ is contained in the convex hull of $A_i KS$. The range of ψ is also contained in S, since for a fixed $u_0 \in S$ and $1 \leq i \leq n$, if $\|A_i K u_0 - v_0\| \geq 1$ then $\beta_i(Ku_0) = 0$ (this follows from the definition of α_i); hence $\|\sum\limits_{i=1}^{n} \beta_i(Ku_0) A_i K u_0 - v_0\| \leq \sum\limits_{i=1}^{n} \beta_i(Ku_0) = 1$.

Note that $\bigcup\limits_{i=1}^{n} \overline{A_i KS}$ is compact. By Mazur's Theorem, the closed convex hull of $\bigcup\limits_{i=1}^{n} \overline{A_i KS}$ is likewise compact. Let $\mathcal{C} = S \cap (\bigcup\limits_{i=1}^{n} \overline{A_i KS})$. Then $\mathcal{C}$ is a compact, convex set for which $\psi(\mathcal{C}) \subset \mathcal{C}$. As mentioned earlier, we can now use the Schauder Fixed Point Theorem to get a $v_1 \in \mathcal{H}$ such that $\sum\limits_{i=1}^{n} \beta_i(Kv_1) A_i K v_1 = v_1$. Therefore, $A = \sum\limits_{i=1}^{n} \beta_i(Kv_1) A_i$ is an operator in $\mathfrak{A}$ for which $AKv_1 = v_1$, so AK has 1 as an eigenvalue. From the comment made at the beginning of the proof this allows us to conclude that the weak closure of $\mathfrak{A}$ contains a finite rank compact operator, and hence is all of $\mathcal{B}(\mathcal{H})$. □

Note in particular from the proof that if $\mathfrak{A}$ satisfies the conditions of the above theorem, then there exists a compact operator in $\mathfrak{A}$ whose spectrum contains 1. This observation will be useful in Chapter 4, when we discuss algebras of compact operators.

For the rest of the section we discuss the relationship between compact operators, triangular operators and quasitriangular operators, as well as methods of identifying quasitriangular operators.

Definition 3.8. *An operator T is* ***triangular*** *if there exists an increasing sequence* *of finite rank projections P_n such that $P_n \to I$ in the strong operator topology, and $\|P_nTP_n - TP_n\| = 0$ for each n.*

From the above definition it is obvious that, for each n, $P_n\mathcal{H}$ is an invariant subspace of T. The closure of the set of triangular operators is the set of quasitriangular operators, as defined below.

Definition 3.9. *An operator T is **quasitriangular** if there is an increasing sequence of finite rank projections P_n such that $P_n \to I$ in the strong operator topology, and $\|P_nTP_n - TP_n\| \to 0$.*

Compact operators are quasitriangular since any compact operator on a Hilbert space can be written as a limit of finite rank operators. However, as we shall show below, it is not the case that all compact operators are triangular.

The Volterra operator $V : L^2(0,1) \to \mathbb{C}$ is defined by $(Vf)(x) = \int_0^x f(y)dy$. Let $\mathcal{M}_\alpha = \{f \in L^2(0,1) : f = 0 \text{ a.e. on } [0,\alpha]\}$. Then $\operatorname{Lat} V = \{\mathcal{M}_\alpha : \alpha \in [0,1]\}$ (for a proof of this, see [18], Theorem 4.14). It is known that V is compact and hence quasitriangular, but V is not triangular. In fact, if we let $\mathfrak{A}_V$ be the unital Banach algebra generated by V, there is no contractive homomorphism which maps V to a triangular operator. The proof of this fact, given below, is due to D. R. Farenick (from a private communication). Before we can present the proof we need to define the numerical range of an operator and present some of its properties.

The **spatial numerical range** of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(T) := \{\langle Tv|v\rangle : v \in \mathcal{H}, \|v\| = 1\}.$$

The Toeplitz-Hausdorff Theorem tells us that the numerical range is a convex set (for a proof of this theorem, see [19]). Halmos mentions in [10] that $W(V)$ is the set lying between the curves $t \mapsto \frac{1-\cos(t)}{t^2} \pm i\frac{t-\sin(t)}{t^2}$ for $0 \le t \le 2\pi$. A calculation shows that $0 \in \partial W(V)$.

Recall that if $\mathfrak{B}$ is a $\mathcal{C}^*$-algebra then the set of states of $\mathfrak{B}$, denoted by $\mathcal{S}(\mathfrak{B})$, consists of the positive linear functionals on $\mathfrak{B}$ which have norm 1. An equivalent description of $\mathcal{S}(\mathfrak{B})$ which will be used later is that it consists of the linear functionals on $\mathfrak{B}$ which have norm 1 and evaluate to 1 at I. The **algebraic numerical range** of $b \in \mathfrak{B}$ is defined by $W_a(b) = \{\Psi(b) : \Psi \in \mathcal{S}(\mathfrak{B})\}$. The extreme points of $\mathcal{S}(\mathfrak{B})$ are called pure states. If $\mathfrak{B}$

is unital then $\mathcal{S}(\mathfrak{B})$ is compact (in the weak* topology) as well as convex, so the Krein-Milman theorem gives us that $\mathcal{S}(\mathfrak{B})$ is the closed convex hull of the pure states. Suppose additionally that $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$. Then another subset of $\mathcal{S}(\mathfrak{B})$ which is of interest is the set of **vector states** of $\mathfrak{B}$, given by $\{\phi_v : \mathfrak{B} \to \mathbb{C} : \phi_v(T) = \langle Tv|v\rangle, v \in \mathcal{H} \text{ and } \|v\| = 1\}$. If $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$ contains the identity operator then the pure states are contained in the weak* closure of the set of vector states ([3], Theorem 12). It follows that $\overline{W(T)} = W_a(T)$.

Suppose $T \in \mathcal{B}(\mathcal{H})$ is an operator and ρ is a contractive homomorphism from $\mathfrak{A}_T$ to $\mathcal{B}(\mathcal{H})$. Then $W_a(\rho(T)) = \{\Psi(\rho(T)) : \Psi(I) = 1, \|\Psi\| = 1\}$. Note that for each $\Psi \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$ we have that $\Psi \circ \rho$ is a linear functional on $\mathfrak{A}_T$ and $(\Psi \circ \rho)(I) = 1$; moreover, since ρ is contractive we also have $\|\Psi \circ \rho\| = 1$. By the Hahn-Banach Theorem $\Psi \circ \rho$ can be extended to a linear functional Φ_Ψ on $\mathcal{B}(\mathcal{H})$ with $\|\Phi_\Psi\| = 1$. Hence $\{(\Psi \circ \rho)(T) : \Psi(I) = 1, \|\Psi\| = 1\} \subset \{\Phi(T) : \Phi(I) = 1, \|\Phi\| = 1\} = W_a(T)$ Thus we have shown that $W_a(\rho(T)) \subset W_a(T)$, which from the comments in the previous paragraph implies $\overline{W(\rho(T))} \subset \overline{W(T)}$.

We now apply this information to $\mathfrak{A}_V$, the unital norm-closed algebra generated by the Volterra operator V. Suppose by contradiction that a contractive homomorphism $\rho : \mathfrak{A}_V \to \mathcal{B}(\mathcal{H})$ such that $\rho(V)$ is triangular did exist. Let $\{u_k\}$ be an orthonormal basis of $\mathcal{H}$ for which $\rho(V)$ is triangular, say $\rho(V) = [t_{ij}]$, where $t_{ij} = 0$ for $i < j$. Now V is quasinilpotent, so $\sigma(V) = \{0\}$. Since $\sigma(\rho(V)) \subset \sigma(V)$ (ρ is a homomorphism), we get that $t_{ii} = 0$ for each i. However, $\rho(V)$ is not the zero operator, so there exist indices $r < s$ such that $t_{rs} \neq 0$. Denote by R the compression of $\rho(V)$ to the subspace spanned by u_r and u_s. Then $R = \begin{bmatrix} t_{rr} & t_{rs} \\ t_{sr} & t_{ss} \end{bmatrix} = \begin{bmatrix} 0 & t_{rs} \\ 0 & 0 \end{bmatrix}$, where $t_{rs} \neq 0$. Note that 0 is an interior point of $W(R)$. Since R is a restriction of $\rho(V)$, we have $W(R) \subset W(\rho(V))$. But $W(\rho(V)) \subset \overline{W(\rho(V))} \subset \overline{W(V)}$ (shown a few paragraphs earlier), contradicting the fact that $0 \in \partial W(V)$.

Theorem 3.10. *Let A be a quasitriangular operator and $\epsilon > 0$. Then we can find T triangular and K compact such that $A = T + K$ and $\|K\| \leq \epsilon$.*

Proof. Since A is quasitriangular we can find an increasing sequence of finite rank projections $\{E_n\}_{n\geq 1}$ such that $AE_n\mathcal{H} \subseteq E_{n+1}\mathcal{H}$ (since AE_n is a finite rank operator), $E_n \to I$ strongly and $\|AE_n - E_nAE_n\| \leq \frac{1}{n^2}$ (the existence of such E_n follows from the definition of quasitriangularity).

Let $\mathcal{H}_i = (E_i - E_{i-1})\mathcal{H}$ for $i \geq 2$ and $\mathcal{H}_1 = E_1\mathcal{H}$. Then with respect to the decomposition $HS = \oplus\mathcal{H}_i$ we get $A = [A_{ij}]$ where $A_{ij} : \mathcal{H}_i \to \mathcal{H}_j$ and $A_{ij} = 0$ for $i \geq j+2$.

Let $A_n = \sum_{i=n}^{\infty} AE_i - E_iAE_i$ (note that the sum converges for each n because of the norm condition above). Clearly $\|A_n\| \to 0$ as $n \to \infty$; let $K = A_N$, where N is chosen such that $\|K\| < \epsilon$. We will show that K is compact and $A - K$ is triangular.

Write $K = [K_{ij}]$ with respect to the above decomposition of $\mathcal{H}$. The definition of K gives us that $K_{i+1,i} = A_{i+1,i}$ for $i \geq N$ and $K_{ij} = 0$ otherwise. Let $K_s = \sum_{j=N}^{s} AE_j - E_jAE_j$ for $s \geq N$. Since E_j is a finite rank projection for each j, it is clear that K_s is a finite rank operator. Moreover $\|K - K_s\| = \|K_{s+1}\| = \|\sum_{i=s+1}^{\infty} AE_i - E_iAE_i\| \leq \sum_{i=s+1}^{\infty} 1/i^2$, which can be made as small as we want by choosing s large enough. Hence $K_s \to K$; so K is a limit of finite rank operators, and as such it is compact. Moreover, for each $m \geq N$ we have $(I - E_m)(A - K)E_m = 0$, so $\{E_m\}_{m \geq N}$ is a sequence of increasing finite rank projections which can be used to show that $(A - K)$ satisfies the definition of a triangular operator. Therefore $A = K + (A - K)$, where $A - K$ is triangular and K is a compact operator with $\|K\| < \epsilon$. □

Definition 3.11. *An operator T is **biquasitriangular** if T and T^* are both quasitriangular.*

Since the set of compact operators is self-adjoint and compact operators are quasitriangular, it follows that compact operators are biquasitriangular. There is a very useful theorem of Apostol, Foias and Voiculescu which enables us to identify quasitriangular operators. Since the proof is quite involved, it is not included here; it can be found in [1]. First, however, we need the following definition.

Definition 3.12. *For an operator $T \in \mathcal{B}(\mathcal{H})$ we say that T is **semi-Fredholm** if $ranT$ is closed and at least one of $nulT = dim \ker T$ and $nulT^* = dim \ker T^*$ is finite.*
*For T semi-Fredholm we define the **Fredholm index** of T to be $ind(T) = nulT - nulT^*$ (with the convention that $ind(T) = \infty$ if $nulT = \infty$, and $ind(T) = -\infty$ if $nulT^* = \infty$).*
Finally, for any operator T we define $\rho_{sF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is semi-Fredholm$\}$.

Theorem 3.13. *Consider $T \in \mathcal{B}(\mathcal{H})$. Then T is quasitriangular if and only if $ind(T - \lambda I) \geq 0$ for all $\lambda \in \rho_{sF}(T)$.*

In particular, the above theorem allows us to conclude that T is biquasitriangular if and only if $\operatorname{ind}(T-\lambda I)=0$ for all $\lambda\in\rho_{sF}(T)$. Normal operators are biquasitriangular because $\ker(N-\lambda I)=\ker(N-\lambda I)^*$ for each $\lambda\in\mathbb{C}$, and hence $\operatorname{ind}(N-\lambda I)=0$ whenever $\lambda\in\rho_{sF}(N)$. Suppose T is an operator similar to N, say $T=S^{-1}NS$. Then $(T-\lambda I)=S^{-1}(N-\lambda I)S$ for any $\lambda\in\mathbb{C}$ and $u\mapsto S^{-1}u$ is a bijection between $\ker(N-\lambda I)$ and $\ker(T-\lambda I)$. So $\operatorname{nul}(T-\lambda I)=\operatorname{nul}(N-\lambda I)$, and similarly $\operatorname{nul}(T-\lambda I)^*=\operatorname{nul}(N-\lambda I)^*$. Thus $\operatorname{ind}(T-\lambda I)=\operatorname{ind}(N-\lambda I)=0$ for any λ, and hence T is itself biquasitriangular. In fact, it can be shown that the set of biquasitriangular operators is the closure of the set of operators similar to a normal operator (see [11]).

3.2 Reductive Algebras

In this section we discuss the properties of an algebra $\mathfrak{A}$ which follow as a result of certain properties of the lattice of invariant subspace of $\mathfrak{A}$.

Definition 3.14. *Consider a Banach algebra $\mathfrak{A}$ and a Hilbert space $\mathcal{H}$ which is a Banach module for $\mathfrak{A}$. We say that $\mathcal{H}$ has the* ***reduction property*** *if for every closed submodule $\mathcal{V}\subseteq\mathcal{H}$ there is another closed submodule $\mathcal{W}\subseteq\mathcal{H}$ with $\mathcal{H}=\mathcal{V}\oplus\mathcal{W}$. If $\mathfrak{A}\subset\mathcal{B}(\mathcal{H})$ with the standard module action on $\mathcal{H}$, we refer to $\mathfrak{A}$ as a* ***reduction algebra****.*
$\mathfrak{A}$ is a ***complete reduction algebra*** *if the module $\mathcal{H}^{(\infty)}$ has the reduction property. In this case we also say that $\mathcal{H}$ has the* ***complete reduction property****.*
$\mathfrak{A}$ is a ***total reduction algebra*** *if every Hilbert space which is an $\mathfrak{A}$-module has the reduction property.*

Consider $\mathfrak{A}$ a total reduction algebra and a Hilbert space $\mathcal{H}$ which is an $\mathfrak{A}$-module. Recall the definition of $\mathcal{H}^{(\infty)}$ from the previous section (before Theorem 3.5). Then $\mathcal{H}^{(\infty)}$ is an $\mathfrak{A}$-module (we can apply the action of $\mathfrak{A}$ to each component), so from the definition of a total reduction algebra it follows that $\mathcal{H}^{(\infty)}$ has the reduction property. Therefore, $\mathfrak{A}$ is a complete reduction algebra. However, as we will see later, there are complete reduction algebras which are not total reduction algebras.

The next theorem restates the definition of a total reduction algebra as a cohomology property. This will allow us to relate total reductivity to the concept of amenability, as defined in the previous chapter.

Theorem 3.15. *An operator algebra $\mathfrak{A}$ has the total reduction property if and only if $H^1(\mathfrak{A}, \mathcal{B}(\mathcal{H})) = 0$ for every representation $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$.*

Proof. Suppose $\mathfrak{A}$ has the total reduction property, and $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a representation. Let $D : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a derivation; we want to show that D is inner. In order to use the total reduction property we have to find a representation of $\mathfrak{A}$ that has an invariant subspace. Define $\phi : \mathfrak{A} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by $a \mapsto \begin{bmatrix} \theta(a) & D(a) \\ 0 & \theta(a) \end{bmatrix}$; then ϕ is a representation of $\mathfrak{A}$ on $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, and $\mathcal{H} \oplus 0$ is a submodule of $\mathcal{H} \oplus \mathcal{H}$. Since $\mathfrak{A}$ has the total reduction property, it follows that $\mathcal{H} \oplus 0$ has a complementary module, say V. Since $V + (\mathcal{H} \oplus 0) = \mathcal{H} \oplus \mathcal{H}$ for each $w \in \mathcal{H}$ there must exist at least one $u \in \mathcal{H}$ for which $u \oplus w \in V$. But also $V \cap (\mathcal{H} \oplus 0) = \{0\}$, so such a u must be unique (if $u_1 \oplus v$ and $u_2 \oplus v$ are both in V, then so is $(u_2 - u_1) \oplus 0)$. Therefore, for each $v \in \mathcal{H}$ there is a unique $u_v \in \mathcal{H}$ such that $u_v \oplus v \in V$. Define $T : \mathcal{H} \to \mathcal{H}$ by $T(v) = u_v$. Then $V = \{Tv \oplus v : v \in \mathcal{H}\}$. From the fact that V is a subspace it follows that T must be linear; and, since V is closed, T is continuous by the Closed Graph Theorem.

Thus $V = \{Tu \oplus u : u \in \mathcal{H}\}$ where T is a continuous operator. But V is invariant for $\phi(a)$, so for any $u \in \mathcal{H}$ there is a $v \in \mathcal{H}$ such that $\begin{bmatrix} \theta(a) & D(a) \\ 0 & \theta(a) \end{bmatrix} \begin{bmatrix} Tu \\ u \end{bmatrix} = \begin{bmatrix} Tv \\ v \end{bmatrix}$, which implies $\begin{bmatrix} \theta(a)Tu + D(a)u \\ \theta(a)u \end{bmatrix} = \begin{bmatrix} Tv \\ v \end{bmatrix}$. By equating matrix entries, $v = \theta(a)u$; so $D(a)u = (T\theta(a) - \theta(a)T)(u)$ for all $u \in \mathcal{H}$, i.e. $D(a) = T\theta(a) - \theta(a)T$. Therefore D is inner, as desired.

Conversely, suppose $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathfrak{A}$. Consider U a submodule of $\mathcal{H}$. We want to use the fact that every derivation from $\mathfrak{A}$ to $\mathcal{B}(\mathcal{H})$ is inner to find a complementary submodule V of U. This is accomplished by reversing the steps from the previous paragraph. For each $a \in \mathfrak{A}$ the matrix $\theta(a)$ has the form $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ with respect to the decomposition $\mathcal{H} = U \oplus U^{\perp}$ (since U is an invariant subspace). Then $\pi : a \mapsto \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ is a representation of $\mathfrak{A}$. Define D by $D(a) = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}$. Note that

$$D(ab) = \begin{bmatrix} 0 & A_{11}B_{12} + A_{12}B_{22} \\ 0 & 0 \end{bmatrix} = \pi(a)D(b) + D(a)\pi(b).$$

Therefore D is a derivation with respect to π. But then by hypothesis D is inner, so there exists a $T \in \mathcal{B}(\mathcal{H})$ such that $D(a) = \pi(a) \cdot T - T \cdot \pi(a)$. Equating the entries in the matrices we get $A_{12} = A_{11}T_{12} - T_{12}A_{22}$ for all $a \in \mathfrak{A}$.

Let $V = \{-Tv \oplus v : v \in U^{\perp}\}$. Then

$$\begin{bmatrix} A_{11} & A_{11}T_{12} - T_{12}A_{22} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} -Tv \\ v \end{bmatrix} = \begin{bmatrix} -TA_{22}v \\ A_{22}v \end{bmatrix},$$

so V is an invariant subspace. Moreover, it is clear that $U \cap V = \{0\}$ and $U + V = \mathcal{H}$. Hence V is a closed submodule which complements U. Therefore $\mathfrak{A}$ has the total reduction property. $\square$

In particular, this result allows us to relate the total reduction property to amenability. Suppose $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a representation. The action of $\mathfrak{A}$ on $\mathcal{B}(\mathcal{H})$ is given by $a \cdot T = \theta(a)T$ and $T \cdot a = T\theta(a)$ for each $a \in \mathfrak{A}$ and $T \in \mathcal{B}(\mathcal{H})$. In order to relate this to amenability, we need to identify $\mathcal{B}(\mathcal{H})$ as a dual space, and show that the dual action of $\mathfrak{A}$ on $\mathcal{B}(\mathcal{H})$ is identical to the one just described. Below we introduce the space C_1, which is the predual of $\mathcal{B}(\mathcal{H})$.

For K a compact operator, denote the eigenvalues of $|K| = (KK^*)^{1/2}$ by $\{s_n\}_{n\in\mathbb{N}}$; we know that $s_n \to 0$. For $1 \le p \le \infty$ the Schatten p-class of operators, denoted by C_p, is defined to consist of those compact operators for which $\{s_n\}_{n\in\mathbb{N}} \in l^p$. We are particularly interested in the situations when $p = \infty$, since C_∞ is clearly the set of all compact operators, and when $p = 1$. For $K \in C_1$, let $\{\phi_n\}$ be an orthonormal basis of $\mathcal{H}$ and define $tr(K) = \sum_{n=1}^{\infty} \langle K\phi_n | \phi_n \rangle$. It can be shown that this sum converges and is independent of the choice of basis ([6]). Since we can define a trace function as described above, the operators in C_1 are called the trace class operators. It is known that C_1 is a two-sided ideal in $\mathcal{B}(\mathcal{H})$. Also, for $A \in \mathcal{B}(\mathcal{H})$ and $K \in C_1$ we have that $tr(AK) = tr(KA)$.

The following two theorems give us the relationship between the trace class operators and $\mathcal{K}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ respectively. The proofs can be found in [6].

Theorem 3.16. *For $T_0 \in C_1$ define the linear functional $\phi_{T_0} : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ by $\phi_{T_0}(K) = tr(T_0K)$. Then $T \mapsto \phi_T$ is an isometric isomorphism from C_1 to $\mathcal{K}(\mathcal{H})^*$.*

Theorem 3.17. *For $S_0 \in \mathcal{B}(\mathcal{H})$ we can define the map $\phi_{S_0} : C_1 \to \mathbb{C}$ given by $\phi_{S_0}(K) = tr(S_0K)$. Then $S \mapsto \phi_S$ is an isometric isomorphism from C_1^* to $\mathcal{B}(\mathcal{H})$.*

So $\mathcal{B}(\mathcal{H})$ is a dual module, and there is a weak* topology on $\mathcal{B}(\mathcal{H})$ (in some of the literature this topology is also called the ultraweak topology, and it should be noted that it coincides with the σ-weak topology). If θ is a representation of $\mathfrak{A}$ on $\mathcal{B}(\mathcal{H})$ then, as explained earlier, the action of $\mathfrak{A}$ on C_1 is given by $a \cdot C = \theta(a)C$ and $C \cdot a = C\theta(a)$. So for $S_0 \in \mathcal{B}(\mathcal{H})$ and $K \in C_1$ we have

$$\begin{aligned} [a \cdot \phi_{S_0}](K) &= \phi_{S_0}(K \cdot a) && \text{(the definition of the dual action)} \\ &= \phi_{S_0}(K\theta(a)) \\ &= tr(S_0 K\theta(a)) \\ &= tr(K\theta(a)S_0) && \text{(properties of trace)} \\ &= \phi_{\theta(a)S_0}(K) \end{aligned}$$

Therefore, $a \cdot \phi_{S_0} = \phi_{\theta(a)S_0}$. Similarly we can show that $\phi_{S_0} \cdot a = \phi_{S_0\theta(a)}$. So we may identify $\mathcal{B}(\mathcal{H})$ with C_1^* and under this identification $a \cdot S_0 = \theta(a)S_0$ and $S_0 \cdot a = S_0\theta(a)$ for $a \in \mathfrak{A}$ and $S_0 \in \mathcal{B}(\mathcal{H})$. Therefore, the dual action of $\mathfrak{A}$ on $\mathcal{B}(\mathcal{H})$ is identical to the module action of $\mathfrak{A}$ on $\mathcal{B}(\mathcal{H})$.

If, moreover, $\mathfrak{A}$ is amenable, then $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule E; so, in particular, since $\mathcal{B}(\mathcal{H})$ is a dual $\mathfrak{A}$-bimodule, $\mathcal{H}^1(\mathfrak{A}, \mathcal{B}(\mathcal{H})) = \{0\}$. Hence it follows that if $\mathfrak{A}$ is amenable, then $\mathfrak{A}$ has the total reduction property. The converse is not in general true. An example of a Banach algebra that has the total reduction property but is not amenable is $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ an infinite dimensional, separable Hilbert space. See [8], Corollary 2.4.7 and the comment following for an explanation of why this is true (the proof relies on several results not covered here).

The cohomology definition of total reduction algebra also allows us to easily show the following.

Theorem 3.18. *Let $\mathfrak{A}$ be a Banach algebra with the total reduction property. Suppose $\mathfrak{B}$ is a Banach algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism such that $\phi(\mathfrak{A})$ is dense in $\mathfrak{B}$. Then $\mathfrak{B}$ has the total reduction property.*

Proof. This proof is identical to the one for Theorem 2.19, except that instead of considering derivations to an arbitrary dual space we consider derivations to the set of bounded operators on a Hilbert space. □

Note however that we cannot replace the total reduction property by the complete reduction property in the above theorem. For example, suppose $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ has the complete reduction property but not the total reduction property. Then by definition there exists a Hilbert space $\mathcal{L}$ and a representation $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{L})$ such that $\mathcal{L}$ does not have the reduction property. It follows that $\mathcal{L}$ does not have the complete reduction property; hence, even though $\mathfrak{A}$ is a complete reduction algebra, $\overline{\theta(\mathfrak{A})} \subset \mathcal{B}(\mathcal{L})$ is not. Therefore, the complete reduction property is not in general preserved by homomorphisms.

Theorem 3.19. *Let $\mathfrak{A}$ be an operator algebra, and $\mathcal{H}$ a Hilbertian $\mathfrak{A}$-module with the complete reduction property. Then there exists $K \geq 1$ such that for any submodule $V \subseteq \mathcal{H}$ there is a module projection $P : \mathcal{H} \to V$ with $\|P\| \leq K$.*

Proof. We will prove the result by contradiction. Suppose that no such K exists. Then for each $i \in \mathbb{N}$ we can find a submodule V_i such that any projection from $\mathcal{H}$ to V_i has norm greater than i.

Consider $U = \sum^{\oplus} V_i$ as a submodule of $\mathcal{H}^{(\infty)}$. Since $\mathfrak{A}$ has the complete reduction property there exists a module W such that $\mathcal{H}^{(\infty)} = U \oplus W$. Hence there is a module projection P onto U. For each $i \in \mathbb{N}$ we define a projection P_i onto V_i as follows: identify $\mathcal{H}$ with the i^{th} copy of it in $\mathcal{H}^{(\infty)}$, apply P, and get the i^{th} coordinate from the result. Since P is a projection and the i^{th} module in the direct sum for U is V_i, we obtain a projection of $\mathcal{H}$ onto V_i. Moreover, it is clear that $\|P_i\| \leq \|P\|$. But by the way V_i was chosen at the beginning of the proof, we also have $\|P_i\| \geq i$ for each i, which leads to a contradiction. Therefore, we can find a K such that for every module V there is a projection P onto V such that $\|V\| \leq K$. □

The minimum K which satisfies the above Theorem is called the **projection constant** of $\mathfrak{A}$.

Lemma 3.20. *Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a complete reduction algebra with projection constant K, and let $\mathcal{P}$ be the set of central projections of $\mathfrak{A}''$. Then $\mathcal{P}$ is bounded by K. Also, there exists a similarity S of $\mathcal{H}$ which makes all the central projections self-adjoint.*

Proof. Consider any $P \in \mathcal{P}$. Then, since P commutes with any $A \in \mathfrak{A}$, $P\mathcal{H}$ is a submodule of $\mathcal{H}$. Hence by Theorem 3.19 we can find a module projection Q onto $P\mathcal{H}$ with $\|Q\| \leq K$. But then, using the fact that P and Q are idempotents it follows that $PQ = Q$ and $QP = P$ (since P and Q have the same range, $P\mathcal{H}$). Thus, since P and Q commute, we get $P = Q$. Therefore $\|P\| \leq K$, as desired.

$\mathcal{P}$ is a set of commuting idempotents which is uniformly bounded by K and closed under symetric difference (it is easy to check that if $P, Q \in \mathcal{P}$, then so is $P + Q - 2PQ$). Hence by Lemma 3.2 we know that we can find a matrix S such that SPS^{-1} is self-adjoint for all $P \in \mathcal{P}$ and $\|S\|\|S^{-1}\| \leq (1+2K)^2$. This proves the second part of the Lemma. $\square$

Definition 3.21. *Suppose V, W are submodules of $\mathcal{H}$ under the action of $\mathfrak{A}$. Then ϕ is a **module map** if $\phi : V \to W$ is a linear map which satisfies $\phi(a \cdot v) = a \cdot \phi(v)$.*

If $\phi : V \to W$ is a non-zero module map, we say that ϕ **intertwines** V and W. It is of course possible that no such non-zero module map exists. As a simple example, consider the algebra $\mathfrak{A} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ acting on $\mathbb{C}^2$. Let $\mathcal{M} = \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathbb{C} \right\}$. Then $\mathcal{M}$ is a submodule of $\mathbb{C}^2$, and the identity map from $\mathcal{M}$ to $\mathbb{C}^2$ is a module map. However, we shall show that there is no non-zero module map from $\mathbb{C}^2$ to $\mathcal{M}$. If such a module map did exist, it would have to be a bounded operator on $\mathbb{C}^2$ whose range is contained in $\mathcal{M}$ and which commutes with all the operators in $\mathfrak{A}$. But $\mathfrak{A}' = \{\lambda I : \lambda \in \mathbb{C}\}$, so there is no non-zero operator in the commutant of $\mathfrak{A}$ which has range contained in $\mathcal{M}$. Therefore, there is no non-zero module map from $\mathbb{C}^2$ to $\mathcal{M}$.

In particular, the above example shows that it is possible to have two modules V and W such that there exists a nonzero module map from V to W but no such map exists from W to V. However, we shall show below that this can no longer occur if $\mathfrak{A}$ is a complete reduction algebra.

Theorem 3.22. *Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a complete reduction algebra. Suppose V, W are submodules of $\mathcal{H}$ such that $\phi : V \to W$ is a non-zero module map. Then there exists a non-zero module map $\psi : W \to V$.*

Proof. If $V \cap W \neq \{0\}$ or $V + W$ is not closed, we can consider the action of $\mathfrak{A}$ on $\mathcal{H} \oplus \mathcal{H}$, embed V into $\mathcal{H} \oplus 0$ and embed W into $0 \oplus \mathcal{H}$. Then $\mathcal{H} \oplus \mathcal{H}$ has the complete reduction property, and there is a correspondence between the nonzero module maps from V to W and those from $V \oplus 0$ to $0 \oplus W$ (and similarly for module maps from W to V). Hence we have found an equivalent question, but $(V \oplus 0) + (0 \oplus W)$ is closed and $(V \oplus 0) \cap (0 \oplus W) = \{0\}$.

Therefore, we can assume without loss of generality that $V \cap W = \{0\}$ and $V + W$ is closed. So V + W is a submodule of $\mathcal{H}$, and as such has the complete reduction property.

By restricting the representation of $\mathfrak{A}$ to V + W we can assume $\mathcal{H} = V \oplus W$. Thus any module map on $\mathcal{H}$ has the form $\begin{array}{cc} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & \begin{array}{c} V \\ W \end{array} \end{array}$ where A is a module map from V to V, B is a module map from W to V, C is a module map from V to W, and D is a module map from W to W. Assume that there is no non-zero module map from W to V. Then we must have that $B = 0$ in the above representation.

By assumption, a module map from V to W exists, so suppose T is such a map. For each $\lambda \in \mathbb{R}^+$ let $M_\lambda = \{u \oplus \lambda Tu : u \in V\}$; note that, since $\begin{bmatrix} I & 0 \\ \lambda T & 0 \end{bmatrix} \in \mathfrak{A}'$ is an idempotent, M_λ is a submodule of $V \oplus W$. We know that a module projection onto M_λ exists; let us figure out what this projection would look like. By the above discussion it must have the form $\begin{array}{cc} \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} & \begin{array}{c} V \\ W \end{array} \end{array}$. Since $\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} u \\ \lambda Tu \end{bmatrix} = \begin{bmatrix} u \\ \lambda Tu \end{bmatrix}$, it follows that $A = I$. Finally, if we look at the kernel of this projection it is $0 \oplus U$ for some $U \subset W$. Since however $V \oplus W = M_\lambda \oplus (0 \oplus U)$, the only possibility is $U = W$. Hence the unique module projection onto M_λ is the projection along W, which has the matrix form $\begin{bmatrix} I & 0 \\ \lambda T & 0 \end{bmatrix}$.

If we let K be the projection constant of $\mathfrak{A}$, Theorem 3.19 tells us that for each submodule of $\mathcal{H}$ there is a module projection with norm at most K. Hence we must have $\left\| \begin{bmatrix} I & 0 \\ \lambda T & 0 \end{bmatrix} \right\| \leq K$ for all $\lambda \in \mathbb{R}^+$. But $\left\| \begin{bmatrix} I & 0 \\ \lambda T & 0 \end{bmatrix} \right\| \to \infty$ as $\lambda \to \infty$, so we have obtained a contradiction. Therefore, a non-zero module map from W to V must exist. □

If $\mathfrak{A}$ has the total reduction property instead of just the complete reduction the result of Theorem 3.19 can be strengthened so that the projection constant for a particular representation does not depend on the representation itself, only on its norm.

Theorem 3.23. *Let $\mathfrak{A}$ be an operator algebra with the total reduction property. There is an increasing function $K : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathfrak{A}$ and $V \subseteq \mathcal{H}$ is a submodule then there is a module projection $p : \mathcal{H} \to V$ such that $\|p\| \leq K(\|\theta\|)$.*

Proof. Suppose that for any $C \in \mathbb{R}^+$ we can find K_C such that whenever θ is a representation with $\|\theta\| \leq C$. Then the projection constant of $\theta(\mathfrak{A})$ is at most K_C. Then the function $K(C) = \sup\limits_{\substack{\theta \text{ with} \\ \|\theta\| \leq C}} \{\text{projection constant of } \theta(\mathfrak{A})\}$ is well-defined and satisfies the requirements. So to prove this theorem by contradiction, we assume that there is some C for which no such K_C exists.

Hence, for each $i \in \mathbb{N}$ we can find a representation θ_i such that $\|\theta_i\| \leq C$, and for each θ_i we can find a module V_i such that all projections onto V_i have norm greater than i. We define a representation θ of $\mathfrak{A}$ to $\mathcal{B}(\oplus H_i)$ by applying θ_i to the i^{th} coordinate. Then $\|\theta\| \leq C$ (since $\|\theta_i\| \leq C$ for each i).

Let $V = \oplus V_i$. We are given that $\mathfrak{A}$ has the total reduction property; hence $\theta(\mathfrak{A})$ has the complete reduction property. This means that there exists a module projection P onto V. As in the proof for Theorem 3.19 we can restrict P to its i^{th} coordinate to obtain a projection P_i onto the module V_i. But then $\|P_i\| < \|P\|$ for all i, and since each V_i was chosen such that any projection onto it has norm greater than i we obtain a contradiction. Therefore, we can construct the desired increasing function K. □

We now briefly discuss $\mathcal{C}^*$-algebras with the total reduction property. Namely, for every representation similar to a *-representation we can show that the similarity matrix has certain restrictions on its norm, as given below.

Theorem 3.24. *Let $\mathfrak{A}$ be a $\mathcal{C}^*$-algebra with the total reduction property, and let $\theta : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a representation which is similar to a *-representation. If K is the projection constant function of Theorem 3.23, then there is a similarity S such that $S\theta S^{-1}$ is a *-representation and $\|S\|\|S^{-1}\| \leq 128K(\|\theta\|)^2$.*

Proof. Suppose $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{G})$ is a *-representation. If S is a similarity such that $\psi = S\theta S^{-1}$, then for $u \in \mathcal{H}$ we have $S(a \cdot u) = S\theta(a)u = S\theta(a)S^{-1}Su = \psi(a)Su = a \cdot (Su)$, so S is a module isomorphism from $\mathcal{H}$ to $\mathcal{G}$. Conversely, if S is a module isomorphism from $\mathcal{H}$ to $\mathcal{G}$ then $\theta = S^{-1}\psi S$.

Let $\alpha = inf\{\|S\|\|S^{-1}\| : S\theta S^{-1}$ is a *-representation$\}$. By assumption $\alpha < \infty$. Scaling S if necessary, we can find a contractive module isomorphism $S : \mathcal{H} \to \mathcal{G}$ such that $S\theta S^{-1} : \mathfrak{A} \to \mathcal{B}(\mathcal{G})$ is a *-representation and $\|S^{-1}\| \leq 2\alpha$.

Consider the representation $\theta \oplus (S\theta S^{-1}) : \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{G})$. Then $\|\theta \oplus (S\theta S^{-1})\| = \|\theta\|$ (since $S\theta S^{-1}$ is a *-homomorphism, and as such it is contractive). Since $\mathfrak{A}$ has the total reduction property, by definition $\mathcal{H} \oplus \mathcal{G}$ has the reduction property. By Theorem 3.23, there exists a constant $M \leq K(\|\theta\|)$ such that for any submodule of $\mathcal{H} \oplus \mathcal{G}$ there exists a projection with norm at most M onto that submodule.

Fix $\mu \in \mathbb{R}$. Then $\mathcal{M} = \{v \oplus \mu S v : v \in \mathcal{H}\}$ is a submodule of $\mathcal{H} \oplus \mathcal{G}$, so there exists a projection P onto $\mathcal{M}$ such that $\|P\| \leq M$. Suppose $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ with respect to $\mathcal{H} \oplus \mathcal{G}$, where each P_i is a module map; from the calculations for $P \begin{bmatrix} 0 \\ u \end{bmatrix} \in \mathcal{M}$, $P \begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathcal{M}$ and $P \begin{bmatrix} v \\ \mu S v \end{bmatrix} = \begin{bmatrix} v \\ \mu S v \end{bmatrix}$ we get that $P_3 = \mu S P_1$, $P_4 = \mu S P_2$ and $P_1 = I - \mu P_2 S$ respectively. Hence P has the form $\begin{bmatrix} I + \mu R S & -R \\ \mu S(I + \mu R S) & -\mu S R \end{bmatrix}$, where $R = -P_2$ from above. Since $\|P\| \leq M$, we must also have $\|\mu S(I + \mu R S)\| \leq M$ and $\|-R\| \leq M$.

Define $T : H \to \mathcal{G} \oplus \mathcal{G}$ by $Tu = \frac{1}{2} S u \oplus \frac{1}{2M}(\mu S(I + \mu R S)u)$. Then from the bounds on the two operators making up T it follows that $\|T\| \leq 1/\sqrt{2}$. Since S is bounded below, T is also bounded below; so T is a contractive module isomorphism onto some closed submodule of $\mathcal{G} \oplus \mathcal{G}$. Hence, by the definition of α, we have $\|T\|\|T^{-1}\| \geq \alpha$. Suppose that for any $u \in \mathcal{H}$ such that $\|u\| = 1$ we had $\|Tu\| > 2\alpha^{-1}$. Then $1 = \|u\| = \|T(T^{-1}u)\| > 2\alpha^{-1}\|T^{-1}u\|$, so since this is true for any $u \in \mathcal{H}$ with $\|u\| = 1$ we get $\|T^{-1}\| < \alpha/2$. Combining this with $\|T\| \leq 1/\sqrt{2}$ we get $\|T\|\|T^{-1}\| < \alpha/(2\sqrt{2}) < \alpha$. This contradiction shows that there is some $u_0 \in \mathcal{H}$ such that $\|u_0\| = 1$ and $\|Tu_0\| \leq 2\alpha^{-1}$.

On the other hand $\|Tu_0\| \geq \frac{1}{2M}\|\mu S(u_0 + \mu R S u_0)\| \geq \frac{1}{2M}\mu\frac{1}{2\alpha}\,|\|u_0\| - \mu\|RSu_0\||$ (note that here we are using the fact that $\|S^{-1}\| \leq 2\alpha$, and so $\|Su_0\| \geq \frac{1}{2\alpha}\|u_0\|$). Hence $\|Tu_0\| \geq \frac{\mu}{4M\alpha}\,|1 - \mu\|RSu_0\||$. Note that if $\|Su_0\| \leq \frac{1}{2M\mu}$ then since $\|R\| \leq M$ we get $\mu\|RSu_0\| \leq \frac{1}{2}$ and $\|Tu_0\| \geq \frac{\mu}{8M\alpha}$. Combining this with $\|Tu_0\| \leq \frac{2}{\alpha}$ we get $\frac{2}{\alpha} \geq \frac{\mu}{8M\alpha}$, which implies $\mu \leq 16M$. Hence if $\mu > 16M$ we have $\|Su_0\| > \frac{1}{2M\mu}$.

Suppose $\mu = 16M + \epsilon$ for some $\epsilon > 0$. Then from the above comment we know $\|Su_0\| > \frac{1}{2M\mu} = \frac{1}{2M(16M+\epsilon)}$. But also $\|Su_0\| \leq 2\|Tu_0\| \leq \frac{4}{\alpha}$ (where the first inequality follows from the definition of T, and the second from the choice of u_0). Thus $\frac{1}{2M(16M+\epsilon)} < \frac{4}{\alpha}$, which implies $\alpha < 8M(16M + \epsilon)$. Since $\epsilon > 0$ is arbitrary, by letting it go to 0 we get

$\alpha < 128M^2$. Therefore $\inf\{\|S\|\|S^{-1}\| : S\theta S^{-1}\text{is a *-representation}\} \leq 128K(\|\theta\|)^2$, giving us the desired result. $\square$

The above theorem is crucial in proving the following:

Theorem 3.25. *[[8], Corollary 2.4.5] Let $\mathfrak{A}$ be a $\mathcal{C}^*$-algebra. Then $\mathfrak{A}$ has the total reduction property if and only if every representation is similar to a *-representation.*

Chapter 4

Operator Algebras Similar to C*-algebras

It has been conjectured that an operator algebra is similar to a $\mathcal{C}^*$-algebra if and only if it has the total reduction property. The results we have so far (for algebras of compact or triangular operators) seem to support this idea, but a definite answer has not been established.

4.1 Algebras of Compact Operators

In this section we consider $\mathfrak{A} \subset \mathcal{K}(\mathcal{H})$. The additional properties of such an algebra will in fact allow us to describe the structure of $\mathfrak{A}$ when it has the complete reduction property. If $\mathfrak{A}$ is such that $\mathfrak{A}''$ has no proper central projections, then $\mathfrak{A}$ is similar to $\mathcal{K}(\mathcal{V})^{(n)}$ for some $\mathcal{V}$ (Theorem 4.9); otherwise $\mathfrak{A}$ is a direct sum of such algebras (Theorem 4.12). The main result of this section is that an algebra of compact operators is similar to a $\mathcal{C}^*$-algebra if and only if it has the complete reduction property.

In order to prove this result we will need the following two theorems:

Theorem 4.1. *[Ringrose] Let K be a compact operator and $\mathcal{C}$ be any maximal nest in $\mathrm{Lat}\, K$. Then the spectrum of K consists of $\{0\}$ and the entries of K at the atoms of $\mathcal{C}$.*

In particular, if $\mathrm{Lat}\, K$ contains a continuous nest, then it contains a maximal nest with no atoms. Hence, by the above theorem, $\sigma(K) = \{0\}$, and K is quasinilpotent. A proof for the following theorem can be found in [8] (see Theorem 4.3.3).

Theorem 4.2. *[Shul'man] Let* $\mathfrak{A} \subset \mathcal{K}(\mathcal{H})$ *be an operator algebra such that Lat*$\mathfrak{A}$ *contains a continuous nest. If* $T = \sum_{i=1}^{n} a_i b_i$ *for some* $n \in \mathbb{N}$, $a_i \in \mathfrak{A}$ *and* $b_i \in \mathfrak{A}'$, *then* T *is quasinilpotent.*

Lemma 4.3. *Let* $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ *be a nondegenerate complete reduction algebra such that* $\mathfrak{A}''$ *has no proper central projections. Then Lat*$\mathfrak{A}$ *contains a non-zero irreducible submodule.*

Proof. We will prove the contrapositive. Suppose $\operatorname{Lat}\mathfrak{A}$ does not contain any irreducible submodules. Then we can use Zorn's Lemma to show that $\operatorname{Lat}\mathfrak{A}$ contains a continuous nest. Denote by $\mathfrak{A} \cdot \mathfrak{A}'$ the algebra generated by products of operators in $\mathfrak{A}$ and $\mathfrak{A}'$. The above theorems of Ringrose and Shul'man (4.1 and 4.2) give us that the operators in $\mathfrak{A} \cdot \mathfrak{A}'$ (and in particular in $\mathfrak{A} \subset \mathfrak{A} \cdot \mathfrak{A}'$) are quasinilpotent.

Since $\mathcal{K}(\mathcal{H})$ is a closed ideal of $\mathcal{B}(\mathcal{H})$, $\mathfrak{A} \cdot \mathfrak{A}'$ is made up of compact quasinilpotent operators. Suppose $\operatorname{Lat}\mathfrak{A} \cdot \mathfrak{A}' = \{\{0\}, \mathcal{H}\}$; then by the comment following Lomonosov's Lemma (Theorem 3.7) we know that there is an operator in $\mathfrak{A} \cdot \mathfrak{A}'$ whose spectrum contains $\{1\}$, contradicting the fact that all the operators in $\mathfrak{A} \cdot \mathfrak{A}'$ are quasinilpotent. Thus $\mathfrak{A} \cdot \mathfrak{A}'$ has a non-trivial invariant subspace, say $\mathcal{M}$.

First we show that $\operatorname{Lat}\mathfrak{A} \cdot \mathfrak{A}' = \operatorname{Lat}\mathfrak{A} \cap \operatorname{Lat}\mathfrak{A}'$. Clearly, $\operatorname{Lat}\mathfrak{A} \cap \operatorname{Lat}\mathfrak{A}' \subset \operatorname{Lat}\mathfrak{A} \cdot \mathfrak{A}'$. To prove the converse, consider $\mathcal{U} \in \operatorname{Lat}\mathfrak{A} \cdot \mathfrak{A}'$. Since $I \in \mathfrak{A}'$ it follows that $\mathcal{U}$ is invariant for any element of $\mathfrak{A}$. Since $\mathfrak{A}$ is a complete reduction algebra there exists a $\mathcal{V} \in \operatorname{Lat}\mathfrak{A}$ such that $\mathcal{H} = \mathcal{U} \oplus \mathcal{V}$. But then $\mathcal{H} = \overline{\mathfrak{A}(\mathcal{H})} = \overline{\mathfrak{A}(\mathcal{U} \oplus \mathcal{V})} \subset \overline{\mathfrak{A}\mathcal{U}} \oplus \overline{\mathfrak{A}\mathcal{V}}$. Since $\mathcal{U}$ is invariant for $\mathfrak{A}$ we have that $\overline{\mathfrak{A}\mathcal{U}} \subset \mathcal{U}$ and hence it follows that $\mathcal{U} = \overline{\mathfrak{A}\mathcal{U}}$. Hence $\mathfrak{A}'\mathcal{U} = \mathfrak{A}'\overline{\mathfrak{A}\mathcal{U}}$. From $(\mathfrak{A} \cdot \mathfrak{A}')\mathcal{U} \subset \mathcal{U}$ we get that $\mathfrak{A}'\overline{\mathfrak{A}\mathcal{U}} \subset \mathcal{U}$, and hence $\mathcal{U} \in \operatorname{Lat}\mathfrak{A}'$. Therefore, $\operatorname{Lat}\mathfrak{A} \cdot \mathfrak{A}' = \operatorname{Lat}\mathfrak{A} \cap \operatorname{Lat}\mathfrak{A}'$.

Hence $\mathcal{M} \in \operatorname{Lat}\mathfrak{A} \cdot \mathfrak{A}'$ implies that $\mathcal{M} \in \operatorname{Lat}\mathfrak{A}$ and $\mathcal{M} \in \operatorname{Lat}\mathfrak{A}'$. Now $\mathfrak{A}$ is a complete reduction algebra, so there exists a $\mathcal{N} \in \operatorname{Lat}\mathfrak{A}$ which complements $\mathcal{M}$. Suppose $P = \begin{array}{c} \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \mathcal{N} \\ \mathcal{M} \end{array}$ is a projection onto $\mathcal{N}$, where $B : \mathcal{M} \to \mathcal{N}$ is a module map. In particular, since B is a module map, $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \in \mathfrak{A}'$. However, $\mathcal{M} \in \operatorname{Lat}\mathfrak{A}'$; so $\left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m \end{bmatrix} : m \in \mathcal{M} \right\} \subset \mathcal{M}$. On the other hand, the range of $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \operatorname{ran} B$ is contained in $\mathcal{N}$. Since $\mathcal{M} \cap \mathcal{N} = \{0\}$, we must have $B = 0$. It follows that $\mathcal{M}$ is the unique complementary module to $\mathcal{N}$, and the only module map from $\mathcal{M}$ to $\mathcal{N}$ is 0. By Theorem 3.22, 0 is the only module map from $\mathcal{N}$ to $\mathcal{M}$ as well.

Thus the elements of $\mathfrak{A}'$ have the form $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ (where $A : \mathcal{N} \to \mathcal{N}$ and $D : \mathcal{M} \to \mathcal{M}$ are module maps). Clearly $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, the projection onto $\mathcal{N}$ along $\mathcal{M}$, commutes with all such matrices, and hence is a proper central projection in $\mathfrak{A}''$.

Therefore, if $\mathfrak{A}''$ contains no proper central projections, then $\operatorname{Lat}\mathfrak{A}$ must contain a non-zero irreducible submodule. □

Lemma 4.4. *Let $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ be a nondegenerate complete reduction algebra, and suppose that $\mathcal{V}, \mathcal{W} \in Lat\mathfrak{A}$. If $\mathcal{V}$ is irreducible and $T : \mathcal{V} \to \mathcal{W}$ is a non-zero module map, then the range of T is closed and T is an isomorphism onto its range.*

Proof. $\overline{T\mathcal{V}}$ is a submodule of $\mathcal{W}$, so $T : \mathcal{V} \to \overline{T\mathcal{V}}$ is a non-zero module map. By Theorem 3.22 there is a module map $S : \overline{T\mathcal{V}} \to \mathcal{V}$ (since $\mathfrak{A}$ is a complete reduction algebra). Note that $ST \in \mathcal{B}(\mathcal{V})$. Moreover, since both S and T are module maps we have $ST(a \cdot v) = a \cdot ST(v)$ for $a \in \mathfrak{A}$ and $v \in \mathcal{V}$. Hence $ST \in \mathfrak{A}|_{\mathcal{V}}'$.

However, since $\mathcal{V}$ is irreducible, $\operatorname{Lat} \mathfrak{A}|_{\mathcal{V}} = \{\{0\}, \mathcal{V}\}$. Moreover, $\mathfrak{A}|_{\mathcal{V}}$ consists of compact operators; so by Lomonosov's Lemma $\mathfrak{A}|_{\mathcal{V}}$ is weakly dense in $\mathcal{B}(\mathcal{V})$. It follows that $\mathfrak{A}|_{\mathcal{V}}' = \mathbb{C}I$. Combining this with the result from the previous paragraph we get that $ST = \alpha I$ for some $\alpha \in \mathbb{C}$. Therefore, we can conclude that the range of T is closed and T is an isomorphism onto its range. □

Lemma 4.5. *Suppose that $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ is a nondegenerate complete reduction algebra. Let $\mathcal{V} \in Lat\mathfrak{A}$ be irreducible and $\mathcal{W} \in Lat\mathfrak{A}$ be arbitrary. There is a non-zero module map $T : \mathcal{W} \to \mathcal{V}$ if and only if $\mathcal{W}$ contains a submodule isomorphic to $\mathcal{V}$.*

Proof. Suppose that there exists a non-zero module map $T : \mathcal{W} \to \mathcal{V}$. Then there is a module map $S : \mathcal{V} \to \mathcal{W}$ by Theorem 3.22, and by the previous theorem $S\mathcal{V}$ is closed and S is an isomorphism onto $S\mathcal{V}$. Hence $S\mathcal{V}$ is a submodule of $\mathcal{W}$ isomorphic to $\mathcal{V}$.

Conversely, suppose $\mathcal{U} \subseteq \mathcal{W}$ is a submodule isomorphic to $\mathcal{V}$. Let $T : \mathcal{U} \to \mathcal{V}$ be the isomorphism. Let $P : \mathcal{W} \to \mathcal{U}$ be the module projection onto $\mathcal{U}$. Then TP is a module map from $\mathcal{W}$ to $\mathcal{V}$. □

Lemma 4.6. *Let $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ be a nondegenerate complete reduction algebra. If $\mathcal{V}, \mathcal{W} \in Lat\mathfrak{A}$ and $\mathcal{V}$ is irreducible, then $\mathcal{V} + \mathcal{W}$ is closed.*

Proof. Since $\mathcal{V}$ is an irreducible module we must have either $\mathcal{V} \cap \mathcal{W} = \{0\}$ or $\mathcal{V} \cap \mathcal{W} = \mathcal{V}$. In the second case, $\mathcal{V} \subseteq \mathcal{W}$, so $\mathcal{V} + \mathcal{W} = \mathcal{W}$ is closed.

Hence, we may suppose that $\mathcal{V} \cap \mathcal{W} = \{0\}$. Now $\mathfrak{A}$ is a complete reduction algebra, so $\mathcal{W}$ has a complement $\mathcal{U} \in \operatorname{Lat}\mathfrak{A}$. Let P be the module projection of $\mathcal{H}$ onto $\mathcal{U}$ with kernel $\mathcal{W}$. Then $P|_{\mathcal{V}}$ is a module map from $\mathcal{V}$ to $\mathcal{U}$, so since $\mathcal{V}$ is irreducible it follows that $P|_{\mathcal{V}}$ is invertible by Theorem 4.4.

If $\mathcal{V} + \mathcal{W}$ is not closed, then $\sup\limits_{\substack{a \in \mathcal{V}, b \in \mathcal{W} \\ \|a\| = \|b\| = 1}} \langle a|b\rangle = 1$ (see [21], theorem 2.1). Hence, we can find $v \in \mathcal{V}$ and $w \in \mathcal{W}$ such that $\|v\| = 1$ and $\|v + w\| = (\langle v + w|v + w\rangle)^{1/2}$ is as small as we want; say $\|v + w\| < (\|(P|_{\mathcal{V}}^{-1})\|\|P\|)^{-1}$. But then we have

$$\begin{aligned} \|v\| &= \|(P|_{\mathcal{V}})^{-1}Pv\| \\ &= \|(P|_{\mathcal{V}})^{-1}P(v + w)\| && (\text{since } w \in \mathcal{W}, \text{ so } Pw = 0) \\ &\leq \|(P|_{\mathcal{V}})^{-1}\|\|P\|\|v + w\| \\ &< 1 && (\text{by choice of } v \text{ and } w). \end{aligned}$$

This contradicts the fact that $\|v\|$ was chosen such that $\|v\| = 1$; thus no such v and w exist, and so $\mathcal{V} + \mathcal{W}$ must be closed. □

Lemma 4.7. *Let $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ be a nondegenerate complete reduction algebra and suppose that $\mathcal{V} \in Lat\mathfrak{A}$ is irreducible. Then $\mathfrak{A}$ contains a projection which restricts to a non-zero projection on $\mathcal{V}$.*

Proof. Consider $\mathfrak{A}|_{\mathcal{V}}$ as a (not necessarily closed) subalgebra of $\mathcal{B}(\mathcal{V})$. Since $\mathcal{V}$ is irreducible, we must have $\operatorname{Lat} \mathfrak{A}|_{\mathcal{V}} = \{\{0\}, \mathcal{V}\}$. The proof to Lomonosov's Lemma (Theorem 3.7) tells us that we can find a compact operator $K \in \mathfrak{A}$ such that $K|_{\mathcal{V}}$ is a compact operator which has 1 as an eigenvalue. Since 1 is an isolated point of the spectrum of K we can use the Riesz functional calculus to find a projection E such that $\sigma(K|_{E\mathcal{H}}) = \{1\}$. Hence E satisfies the requirements of the lemma. □

Lemma 4.8. *Let $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ be an operator algebra with the complete reduction property and suppose that $\mathcal{V} \in Lat\mathfrak{A}$ is an irreducible submodule. Let $\mathcal{F}$ be a family of submodules of $\mathcal{H}$ where each submodule is module isomorphic to $\mathcal{V}$. Let $M = \overline{span} \bigcup\limits_{\mathcal{U} \in \mathcal{F}} \mathcal{U}$. Then M is the direct sum of finitely many submodules isomorphic to $\mathcal{V}$.*

Proof. Construct a sequence $\mathcal{V}_i$ of submodules isomorphic to $\mathcal{V}$ as follows: assume we have a sequence $\{\mathcal{V}_1, \dots \mathcal{V}_n\}$ for $n \geq 1$ (the first module picked, $\mathcal{V}_1$, can be any module in $\mathcal{F}$). If $M \neq \bigoplus_{i=1}^n \mathcal{V}_i$, then we can find $\mathcal{V}_{n+1}$ in $\mathcal{F}$ such that $\mathcal{V}_{n+1} \not\subseteq \bigoplus_{i=1}^n \mathcal{V}_i$. In fact, since all the $\mathcal{V}_i$'s are irreducible, we must have $\mathcal{V}_{n+1} \cap \mathcal{V}_i = \{0\}$ for each $1 \leq i \leq n$. Moreover, $\mathcal{V}_1 + \mathcal{V}_2 + \ldots + \mathcal{V}_{n+1}$ is closed by Theorem 4.6. Thus $\bigoplus_{i=1}^n \mathcal{V}_i + \mathcal{V}_{n+1} = \bigoplus_{i=1}^{n+1} \mathcal{V}_i$.

In this manner we construct a sequence of modules $\{\mathcal{V}_i\}_{i\in\mathbb{N}} \subset \mathcal{F}$ such that $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \ldots \subseteq M$. By Theorem 4.7 we can find $P \in \mathfrak{A}$ a projection such that $P|_{\mathcal{V}}$ is non-zero. But then since each $\mathcal{V}_i$ in the above list is module isomorphic to $\mathcal{V}$, it follows that $P|_{\mathcal{V}_i}$ is also non-zero. Since $\mathfrak{A}$ is an algebra of compact operators, P is compact; in particular, since P is a projection, it must have finite rank; hence our list of submodules $\mathcal{V}_i$ can contain only finitely many elements. It follows that there is some N such that $M = \bigoplus_{i=1}^N \mathcal{V}_i$. $\square$

Theorem 4.9. *Let $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ be a complete reduction algebra, and suppose $\mathfrak{A}''$ contains no proper central idempotents. Then there exists an irreducible submodule $\mathcal{V} \in Lat\mathfrak{A}$, and $\mathfrak{A}$ is similar to $\mathcal{K}(\mathcal{V})^{(n)}$ for some $n \in \mathbb{N}$.*

Proof. By Theorem 4.3, since $\mathfrak{A}''$ does not contain any proper central idempotents, we know that $\mathcal{H}$ contains an irreducible submodule $\mathcal{V}$. Let $\mathcal{F} = \{\mathcal{M} : \mathcal{M}$ is isomorphic to $\mathcal{V}\}$. Lemma 4.8 tells us that the closed span of all modules in $\mathcal{F}$ can be written as $\mathcal{W} := \bigoplus\limits_{i=1}^{n} \mathcal{V}_i$ for some n and modules $\mathcal{V}_i$ isomorphic to $\mathcal{V}$. Since $\mathfrak{A}$ is a complete reduction algebra, we can write $\mathcal{H} = \mathcal{W} \oplus \mathcal{U}$, where the module $\mathcal{U}$ has no submodule isomorphic to $\mathcal{V}$. Since $\mathcal{U}$ has no submodule isomorphic to $\mathcal{V}$, Lemma 4.5 tells us that there is no non-zero map from $\mathcal{U}$ to $\mathcal{V}$. But then there can be no non-zero map from $\mathcal{V}$ to $\mathcal{U}$ either (Theorem 3.22).

So with respect to the decomposition $\mathcal{H} = \mathcal{W} \oplus \mathcal{U}$, the matrices for elements of $\mathfrak{A}'$ look like $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, where $A : \mathcal{W} \to \mathcal{W}$ and $D : \mathcal{U} \to \mathcal{U}$ are module maps. It follows that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, the projection onto $\mathcal{W}$ along $\mathcal{U}$, is a central projection of $\mathfrak{A}''$. But we know that $\mathfrak{A}''$ contains no proper central idempotents; hence $\mathcal{U}$ must in fact be $\{0\}$.

Therefore, $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{V}_i$. For each i, let $T_i : \mathcal{V}_i \to \mathcal{V}$ be a module isomorphism. Then the norm on $\mathcal{H}$ given by $\|\!| \sum\limits_{i=1}^{n} v_i |\!\| = (\sum\limits_{i=1}^{n} \|T_i v_i\|^2)^{1/2}$ is equivalent to the usual norm on $\mathcal{H}$. This renorming gives us a similarity under which $\mathfrak{A}$ is similar to $\mathcal{K}(\mathcal{V})^{(n)}$. $\square$

Lemma 4.10. *Suppose that $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ is a complete reduction algebra such that $\mathfrak{A}''$ contains no proper central projections, and suppose $\mathcal{H}^{(\infty)}$ has projection constant M. Then there is a similarity S on $\mathcal{H}$ with $\|S\|\|S^{-1}\| \leq 128M^2$ such that $S\mathfrak{A}S^{-1}$ is self-adjoint.*

Proof. By Theorem 4.9, since $\mathfrak{A}''$ contains no proper central projections, there is an irreducible module $\mathcal{V} \in Lat\mathfrak{A}$ such that $\mathfrak{A}$ is similar to $\mathcal{K}(\mathcal{V})^{(n)}$ for some $n \in \mathbb{N}$, and $\mathcal{H}$ is module isomorphic to $\bigoplus_{i=1}^{n} \mathcal{V}$.

Let $\alpha = inf\{\|S\|\|S^{-1}\| : S : \mathcal{H} \to \mathcal{V}^{(n)}$ is a module isomorphism$\}$. Since $\mathcal{H} \oplus \mathcal{V}^{(n)}$ is embedded isometrically in $\mathcal{H}^{(n+1)}$, it has the reduction property, and projection constant at most M.

This proof is very similar to the proof of Theorem 3.24, where $\mathcal{G}$ is replaced by $\mathcal{V}^{(n)}$. As before we choose an isomorphism S, fix $\mu \in \mathbb{R}$, let P be a projection onto $\{v \oplus \mu Sv : v \in \mathcal{H}\}$, and define $T : \mathcal{H} \to \mathcal{V}^{(n)} \oplus \mathcal{V}^{(n)}$ by $Tu = \frac{1}{2}Su \oplus \frac{1}{2M}(\mu S(1 + R\mu S)u)$. In order to be able to finish the proof as in Theorem 3.24 we need to show T is in fact a module isomorphism from $\mathcal{H}$ to $\mathcal{V}^{(n)}$. However, this follows easily since T is a contractive module isomorphism onto some closed submodule of $\mathcal{V}^{(n)} \oplus \mathcal{V}^{(n)}$. The submodules of $\mathcal{V}^{(2n)}$ are $\mathcal{V}^{(i)}$ for $i \leq 2n$ (since $\mathcal{V}$ is irreducible), so a submodule module isomorphic to $\mathcal{V}^{(n)}$ is isometrically isomorphic to $\mathcal{V}^{(n)}$. Hence $T : \mathcal{H} \to \mathcal{V}^{(n)}$ is a module isomorphism, so $\|T\|\|T^{-1}\| \geq \alpha$, and result follows as before. □

We have found a description for the structure of a complete reduction algebra $\mathfrak{A} \subset \mathcal{K}(\mathcal{H})$ when its double commutant contains no proper central projections. This suggests that for a general complete reduction algebra $\mathfrak{B} \subset \mathcal{K}(\mathcal{H})$ we should examine the central projections in $\mathfrak{B}''$ and use them to relate $\mathfrak{B}$ to complete reduction algebras whose double commutants do not contain proper central projections.

In general, a von Neumann algebra is generated by its projections ([18], Theorem 7.3). However, we are going to be looking at the von Neumann algebra generated by the central projections of $\mathfrak{B}''$, which has the added property that it is abelian. Moreover, this von Neumann algebra also commutes with the algebra of compact operators $\mathfrak{B} \subset \mathfrak{B}''$; this will enable us to use the result shown below.

Lemma 4.11. *Suppose that $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ is an algebra of compact operators acting non-degenerately on $\mathcal{H}$. If R is an abelian von Neumann algebra commuting with $\mathfrak{A}$, then R*

is generated as a von Neumann algebra by its minimal projections.

Proof. Let $\mathfrak{B}$ be the $\mathcal{C}^*$-algebra generated by $\mathfrak{A}$. Since R is self-adjoint, it commutes with every A^* for $A \in \mathfrak{A}$ as well as with $\mathfrak{A}$, and hence R commutes with $\mathfrak{B}$. So $R \subset \mathfrak{B}'$.

The crucial point in this proof is the structure theory of compact operators; namely, since $\mathfrak{B}$ is a non-degenerate $\mathcal{C}^*$-algebra of compact operators, it is unitarily equivalent to $\{\sum_{\gamma}^{c_0} K_\gamma^{(n_\gamma)} : K_\gamma \in \mathcal{K}(\mathcal{H}_\gamma)\}$ (see [5], Theorem 16.18). This allows us to identify the commutant of $\mathfrak{B}$ as well as certain abelian subalgebras of the commutant. Clearly, if $\mathfrak{B}$ has the form described above, then $\mathfrak{B}'$ is isomorphic to $\sum_\gamma^{l^\infty} \mathbb{M}_{n_\gamma}$. Note that for each γ the projection onto $\mathbb{M}_{n_\gamma}$ is a central projection of $\mathfrak{B}'$.

Now R is contained in a maximal abelian self-adjoint subalgebra of $\mathfrak{B}'$, say M. The maximality of M gives us that $M = \sum_\gamma^{l^\infty}(M \cap \mathbb{M}_{n_\gamma})$, where $M \cap \mathbb{M}_{n_\gamma}$ is a maximal abelian self-adjoint subalgebra of $\mathbb{M}_{n_\gamma}$. But the maximal abelian self-adjoint algebras of $\mathbb{M}_n$ for $n \in \mathbb{N}$ are precisely the subalgebras of $\mathbb{M}_n$ whose matrices are diagonal relative to some fixed orthonormal basis for $\mathbb{C}^n$. Hence the maximal abelian self-adjoint algebras of $\mathbb{M}_n$ are isomorphic to $l^\infty(n)$. Therefore, $M \cong \sum^{l^\infty} l^\infty(n_\gamma)$; by reindexing we can find a set ω such that $M \cong l^\infty(\omega)$.

Therefore, R is a self-adjoint subalgebra of $l^\infty(\omega)$. In fact, we can show that R is isomorphic to $l^\infty(\Lambda)$ for a suitably constructed Λ. Define an equivalence relation on ω by $w_1 \sim w_2$ if and only if $r(w_1) = r(w_2)$ for all $r \in R$. Let Λ be the set of equivalence classes of ω with respect to this relation. Then R is a subalgebra of $l^\infty(\Lambda)$.

Fix $\lambda \in \Lambda$ and define $U_\lambda = \{p \in R : p \text{ is idempotent and } p(\lambda) = 1\}$. The infimum of U_λ is given by a characteristic function on a subset of Λ which contains λ. But R is generated by its projections and for any $\mu \in \Lambda$ with $\mu \neq \lambda$ there exists an $r \in R$ such that $r(\mu) \neq r(\lambda)$ (by the definition of Λ). It follows that the infinum of U_λ is given by χ_λ. Hence $R = l_\infty(\Lambda)$. Therefore, R is generated by its minimal projections, $\{\chi_\lambda : \lambda \in \Lambda\}$. □

Theorem 4.12. *Suppose $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ is a nondegenerate complete reduction algebra. Denote the set of minimal central projections of $\mathfrak{A}''$ by $\mathcal{P}$. For each $P \in \mathcal{P}$ the algebra $\mathfrak{A}_P = P\mathfrak{A}$ is a closed two-sided ideal of $\mathfrak{A}$, and $\mathfrak{A} \cong \sum_{P\in\mathcal{P}}^{c_0} \mathfrak{A}_P$. Moreover, considering $\mathfrak{A}_P$ as a subset of $\mathcal{B}(P\mathcal{H})$, the bicommutant $\mathfrak{A}_P'' \subseteq \mathcal{B}(P\mathcal{H})$ contains no proper central projections.*

Proof. Without loss of generality we can assume that the central projections of $\mathfrak{A}''$ are self-adjoint (otherwise we can apply a similarity to make them self-adjoint by Lemma 3.20).

Fix $P \in \mathcal{P}$, we want to show that $\mathfrak{A}_P$ is a closed two-sided ideal of $\mathfrak{A}$. Suppose that there was some $A \in \mathfrak{A}$ such that $PA \notin \mathfrak{A}$. Since PA is a compact operator and $\mathfrak{A}$ is closed, by the Hahn-Banach Separation Theorem we can find a linear functional ϕ in $\mathcal{K}(\mathcal{H})^* = C_1(\mathcal{H})$ such that $\phi|_{\mathfrak{A}} = 0$ and $\phi(pA) = 1$. Since $\mathfrak{A}''$ is the closure of $\mathfrak{A}$ in the weak* topology of $\mathcal{B}(\mathcal{H})$ and $PA \in \mathfrak{A}''$ we can find a net $(B_\alpha)_\alpha$ in $\mathfrak{A}$ such that $B_\alpha \overset{wk*}{\to} PA$. Note that $\phi(B_\alpha) = 0$ for any α and $\phi(PA) = 1$ by definition. But ϕ is continuous in the weak* topology so we should also have $\phi(B_\alpha) \to \phi(PA)$, a contradiction. This shows that $PA \in \mathfrak{A}$ for each $A \in \mathfrak{A}$. Since P commutes with $\mathfrak{A}$ it follows that $\mathfrak{A}_P$ is a two-sided ideal of $\mathfrak{A}$. To see that $\mathfrak{A}_P$ is also norm closed consider $PA_n \in \mathfrak{A}_P$ converging to some $B \in \mathfrak{A}$. Then given $\epsilon > 0$ there exists an N such that for $n \geq N$ we have $\|PA_n u - Bu\| < \epsilon\|u\|$ for any $u \in \mathcal{H}$. In particular, if we substitute $u = Pv$ for $v \in \mathcal{H}$ we get $\|PA_nPv - BPv\| < \epsilon\|Pv\|$, which means, since P is a projection which commutes with $\mathfrak{A}$, that $\|PA_nv - PBv\| < \epsilon\|v\|$ for any $v \in \mathcal{H}$ and $n \geq N$. Therefore, $PA_n \to PB$, whence $B = PB \in \mathfrak{A}_P$. Therefore, $\mathfrak{A}_P$ is closed, as claimed.

Suppose $Q \in \mathfrak{A}_P''$ is a central projection of $\mathfrak{A}_P''$. Since P is a central projection of $\mathfrak{A}''$ we have $\mathfrak{A}_P'' = P\mathfrak{A}''$ (see [5], Proposition 43.8). It then easily follows that QP is central for $\mathfrak{A}''$. But P and Q are projections, so we also have $0 \leq QP \leq P$. By hypothesis P is minimal as a central projection in $\mathfrak{A}''$, hence either $Q = 0$ or $Q = P$. Therefore, $\mathfrak{A}_P''$ has no proper central projections.

Let $\mathcal{R}$ be the abelian von Neumann algebra generated by the central projections of $\mathfrak{A}''$. In particular $\mathcal{R}$ commutes with $\mathfrak{A} \subset \mathfrak{A}''$, so by Theorem 4.11, $\mathcal{R}$ is generated by its minimal projections. For each $P \in \mathcal{P}$ let $\mathcal{H}_P = P\mathcal{H}$; since $\mathfrak{A}$ is non-degenerate, $\mathcal{H} = \sum^{\oplus} \mathcal{H}_P$ and $\sum\limits_{P \in \mathcal{P}} P = I$ (where the sum is defined using convergence in the strong topology).

We know that $\mathfrak{A}_P$ is an ideal of $\mathfrak{A}$ for each $P \in \mathcal{P}$; also, $\mathcal{P}$ consists of self-adjoint and mutually orthogonal projections. Hence we can embed $\sum\limits_{P \in \mathcal{P}}{}^{c_{00}} \mathfrak{A}_P$ (the algebraic direct sum with finitely many non-zero terms) isometrically into $\mathfrak{A}$. It follows that the norm closure of $\sum\limits_{P \in \mathcal{P}}{}^{c_{00}} \mathfrak{A}_P$, that is $\sum\limits_{P \in \mathcal{P}}{}^{c_0} \mathfrak{A}_P$, is contained in $\mathfrak{A}$. The other inclusion follows because for each $A \in \mathfrak{A}$ we have $A = \sum\limits_{P \in \mathcal{P}} PA$, and A is a compact operator so $\{\|PA\|\}_{P \in \mathcal{P}} \in c_0(\mathcal{P})$. Hence, $\mathfrak{A} = \sum\limits_{P \in \mathcal{P}}{}^{c_0} \mathfrak{A}_P$ as desired. $\square$

Theorem 4.13. *Let $\mathfrak{A} \subseteq \mathcal{K}(\mathcal{H})$ be an operator algebra. Then $\mathfrak{A}$ has the complete reduction property if and only if $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra.*

Proof. Any self-adjoint algebra has the complete reduction property. Since the complete reduction property is preserved by similarities, it follows that if $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra, then $\mathfrak{A}$ has the complete reduction property.

Suppose conversely that $\mathfrak{A}$ has the complete reduction property. We can assume without loss of generality that $\mathfrak{A}$ is not degenerate (otherwise we can restrict $\mathcal{H}$ to $\overline{\mathfrak{A}\mathcal{H}}$). Let $\mathcal{P}$ be the set of minimal projections of $\mathfrak{A}''$. Fix $P \in \mathcal{P}$. We already know that $P\mathfrak{A}$ is an algebra (from Theorem 4.12) which contains no proper central projections. Also if $\mathcal{M}$ is an invariant subspace of $P\mathfrak{A}$, then $P\mathcal{M}$ is an invariant subspace of $\mathfrak{A}$ so since $\mathfrak{A}$ is a complete reduction property there is a subspace $\mathcal{N} \in \mathfrak{A}$ such that $\mathcal{H} = P\mathcal{M} \oplus \mathcal{N}$. Then $P\mathcal{N} \in \operatorname{Lat} P\mathfrak{A}$ and $P\mathcal{H} = P\mathcal{M} \oplus P\mathcal{N}$. Thus $P\mathfrak{A}$ has the complete reduction property. By Theorem 4.10 there is a similarity S_p such that $S_P\, P\mathfrak{A}\,(S_P)^{-1}$ is self-adjoint and $\|S_P\|\|S_P^{-1}\| \leq 128 M_P^2$, where M_P is the projection constant of $(P\mathcal{H})^{(\infty)}$. By scaling if necessary we can ensure $\|S_P\| = 1$, which implies $\|S_P^{-1}\| \leq 128 M_P^2$. Note that if M is the projection constant of $\mathcal{H}^{(\infty)}$, then $M_P \leq M$. Let $S = \oplus_{P \in \mathcal{P}} S_P$. Then $\|S\|\|S^{-1}\| \leq 128 M^2$. Since $\mathfrak{A} \cong \sum\limits_{P \in \mathcal{P}}{}^{c_0} P\mathfrak{A}$ (Theorem 4.12) we get that $S\mathfrak{A}S^{-1}$ is a $\mathcal{C}^*$-algebra. Therefore, $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra, as desired. $\square$

Suppose $\mathfrak{A} \subset \mathcal{K}(\mathcal{H})$ has the complete reduction property. Then by the above theorem $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra $\mathfrak{B}$. Moreover, since the compact operators form an ideal of $\mathcal{B}(\mathcal{H})$, $\mathfrak{B}$ is also an algebra of compact operators. But then, as mentioned previously, $\mathfrak{B}$ is unitarily equivalent to $\sum\limits_{\gamma}^{c_0} \mathcal{K}(\mathcal{H}_\lambda)^{(n_\lambda)}$; hence, $\mathfrak{B}$ is amenable. In the previous chapter we showed that all amenable operator algebras have the total reduction property. So $\mathfrak{B}$ has the total reduction property, and therefore, so does $\mathfrak{A}$. Hence for algebras of compact operators the total reduction property and the complete reduction property are the same. Note that this also implies that every representation of a complete reduction algebra $\mathfrak{A} \subset \mathcal{K}(\mathcal{H})$ is similar to a *-representation.

4.2 Algebras of Triangular Operators

Let $\mathcal{H}$ be a separable Hilbert space and $\{e_n\}$ be a basis for $\mathcal{H}$. We denote by $\mathcal{T}_\infty$ the set of operators which are upper triangular with respect to this basis; in other words, $\mathcal{T}_\infty = \{T \in \mathcal{B}(\mathcal{H}) : \langle Te_j | e_i \rangle = 0 \text{ for } i > j\}$. In this section we show that $\mathfrak{A} \subset \mathcal{T}_\infty$ is similar to an abelian $\mathcal{C}^*$-algebra if and only if it has the total reduction property (the proofs are adapted from [13]).

We will ocasionally find it easier to assume that $\mathfrak{A}$ contains the identity operator. However, this will not result in any loss of generality due to the following theorem.

Theorem 4.14 ([8], Theorem 3.3.6)**.** *An algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ has the total reduction property if and only if the algebra generated by $\mathfrak{A} \cup \{I\}$ does.*

The first theorem we prove establishes that any discussion of total reduction algebras of $\mathcal{T}_\infty$ is necessarily confined to abelian algebras. We then present some general results about abelian total reduction algebras, in particular that every operator in such an algebra has at most countably many eigenvalues, and that we can then find a set of eigenvectors which span the whole space. This will allow us to establish the desired result.

Theorem 4.15. *Suppose that $\mathfrak{A} \subseteq \mathcal{T}_\infty$ has the total reduction property. Then $\mathfrak{A}$ is abelian.*

Proof. Define π_n by $\pi_n(T) = P_n T P_n$ where P_n is the projection onto the span of $\{e_1, \dots e_n\}$. First we will show that $\pi_n(\mathfrak{A})$ is abelian. Now π_n is a homomorphism and $\pi_n(\mathfrak{A})$ is closed (since $\pi_n(\mathfrak{A})$ is finite dimensional); so by Theorem 3.18, $\pi_n(\mathfrak{A})$ also has the total reduction property. However, $\pi_n(\mathfrak{A})$ consists of finite rank operators; since finite rank operators are compact, by Theorem 4.13 there exists a similarity matrix S such that $S^{-1}\pi_n(\mathfrak{A})S$ is a $\mathcal{C}^*$-algebra. For every $R \in \pi_n(\mathfrak{A})$, since R is triangular it has $\mathcal{H}_k = span\{e_1, \dots, e_k\}$ as an invariant subspace. It then follows that $S^{-1}\mathcal{H}_k$ is an invariant subspace of $S^{-1}RS$. It is a well known property of $\mathcal{C}^*$-algebras that if $\mathcal{M}$ is an invariant subspace, then so is $\mathcal{M}^\perp$. Combining this with the fact that $rank\, S^{-1}P_k = k$ we get that $S^{-1}\pi_n(\mathfrak{A})S$ consists of diagonal operators, and as such it is abelian. Therefore, $\pi_n(\mathfrak{A})$ must itself be abelian.

Consider $M, N \in \mathfrak{A}$. Pick any $v \in \mathcal{H}$, and any $\epsilon > 0$. Since $P_n \overset{SOT}{\rightarrow} I$ we can find r such that $\|P_r v - v\| < \epsilon$. So we have

$$\begin{aligned}
\|MNv - NMv\| &= \|MNv - MNP_rv + MNP_rv \\
&\qquad -NMP_rv + NMP_rv - NMv\| \\
&\leq \|MNv - MNP_rv\| + \|MNP_rv - NMP_rv\| + \\
&\qquad \|NMP_rv - NMv\| \\
&\leq \|MN\|\|v - P_rv\| + \|M_rN_rv - N_rM_rv\| + \|NM\|\|P_rv - v\| \\
&\qquad (MNP_r = M_rN_r \text{ since M, N are triangular}) \\
&\leq \|MN\|\epsilon + 0 + \|NM\|\epsilon \\
&\qquad (M_rN_r = N_rM_r \text{ since } \pi_r(\mathfrak{A}) \text{ is abelian})
\end{aligned}$$

Since M and N are fixed (and hence $\|MN\|$ and $\|NM\|$ are constants), and $\epsilon > 0$ is arbitrary it follows that $\|MNv - NMv\| = 0$. But $v \in \mathcal{H}$ was also arbitrary; hence for any M and N in $\mathfrak{A}$ we can show that $MN = NM$. Therefore, $\mathfrak{A}$ is abelian as claimed. $\square$

So all the total reduction algebras in $\mathcal{T}_\infty$ are abelian. For this reason, in this section we are mainly concerned with abelian algebras that have the total reduction property. This additional property enables us to draw some conclusions about the invariant subspaces of the algebra.

Suppose that $\mathfrak{A}$ is an abelian Banach algebra and $T \in \mathfrak{A}$. Then we can show that $\ker T$ and $\overline{\operatorname{ran} T}$ are in $\operatorname{Lat}\mathfrak{A}$. Clearly, both sets are closed subspaces of $\mathcal{H}$, so we only need to check that they are invariant for any operator in $\mathfrak{A}$. Pick any $S \in \mathfrak{A}$; since $\mathfrak{A}$ is abelian, $ST = TS$. For $u \in \ker T$ we have $TSu = STu = S(0) = 0$, so $Su \in \ker T$. Therefore, $\ker T$ is an invariant subspace for $\mathfrak{A}$. For $v \in \operatorname{ran} T$, say $v = Tw$ for some $w \in \mathcal{H}$, we have $Sv = STw = TSw$, and so $Sv \in \operatorname{ran} T$. Hence $Sv \in \operatorname{ran} T$. Using convergent sequences we can then show that $\overline{\operatorname{ran} T}$ is invariant for $\mathfrak{A}$. In particular, if $I \in \mathfrak{A}$, then we can use the above to conclude that, for any $T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, $\ker T - \lambda I$ and $\overline{\operatorname{ran} T - \lambda I}$ are invariant subspaces for $\mathfrak{A}$.

We now consider the question of what an abelian total reduction operator algebra might look like in general. Suppose we have an abelian algebra $\mathfrak{A}$ similar to a $\mathcal{C}^*$-algebra $\mathfrak{B}$. Then $\mathfrak{B}$ is itself abelian, and since $\mathfrak{B}$ is self-adjoint it follows that every operator in $\mathfrak{B}$ is normal. Thus every operator in $\mathfrak{A}$ is similar to a normal operator. Recall from Chapter 3 that the set $\{S^{-1}NS : N \text{ normal }\}$ is dense in the set of biquasitriangular operators. Therefore, if similarity to a $\mathcal{C}^*$-algebra is equivalent to having the total reduction property we would expect that an abelian total reduction algebra consists of biquasitriangular operators. This is the result we prove below.

Theorem 4.16. *Let $\mathfrak{A}$ be a unital, abelian, total reduction subalgebra of $\mathcal{B}(\mathcal{H})$. Then every element of $\mathfrak{A}$ is biquasitriangular.*

Proof. Fix $T \in \mathfrak{A}$. We will use Theorem 3.13 to show that T is biquasitriangular. That is, we want to show that for any $\lambda \in \rho_{sF}(T)$ we have $\operatorname{ind}(T-\lambda I) = 0$.

Fix λ in $\mathbb{C}$. Let $\mathcal{M} := \ker T - \lambda I$. We have shown that $\mathcal{M} \in \operatorname{Lat}\mathfrak{A}$. Write the matrices of $\mathfrak{A}$ with respect to $\mathcal{M}$ and $\mathcal{M}^\perp$. Then there is a similarity matrix S such that $S^{-1}AS = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$ for $A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} \in \mathfrak{A}$ (see Remark 4.17). In particular, since $T|_{\mathcal{M}} = \lambda I$, entry T_1 in the matrix for T is λI. It follows that

$$S^{-1}(T-\lambda I)S = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 0 & (T-\lambda I)|_{\mathcal{M}^\perp} \end{bmatrix} & \begin{array}{c} \mathcal{M} \\ \mathcal{M}^\perp \end{array} \end{array}.$$

From this matrix representation, it is easy to see that $\mathcal{M} \subset \ker(S^{-1}(T-\lambda I)S)^*$. Hence,

$$\begin{aligned} \dim\ker(T-\lambda I)^* &= \dim\ker(S^{-1}(T-\lambda I)S)^* && \text{(since } S \text{ is a bijection)} \\ &\geq \dim \mathcal{M} && \text{(since } M \subset \ker(S^{-1}(T-\lambda I)S)^*) \\ &= \dim\ker(T-\lambda I) && \text{(by definition of } \mathcal{M}) \end{aligned}$$

Therefore, $\operatorname{nul}(T-\lambda I)^* \geq \operatorname{nul}(T-\lambda I)$.

Consider $\mathcal{N} = \overline{\operatorname{ran}(T-\lambda I)}$. Again, we have shown that $\mathcal{N} \in \operatorname{Lat}\mathfrak{A}$. If we write the matrices in $\mathfrak{A}$ with respect to $\mathcal{N}$ and $\mathcal{N}^\perp$, then there is a similarity matrix R such that $R^{-1}AR = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$ for $A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} \in \mathfrak{A}$ (see Remark 4.17). Since $\ker(T-\lambda I)^* = \mathcal{N}^\perp$, it follows that entry T_4 of the matrix for T is λI. Hence

$$R^{-1}(T-\lambda I)R = \begin{array}{cc} \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} \mathcal{N} \\ \mathcal{N}^\perp \end{array} \end{array}.$$

From this matrix representation we can see that $\operatorname{ran}(R^{-1}(T-\lambda I)R)^* \subseteq \mathcal{N}$. Thus we have

$$\begin{aligned} \dim\ker(T-\lambda I) &= \dim\ker(R^{-1}(T-\lambda I)R) && \text{(since } R \text{ is a bijection)} \\ &= \dim\left[\operatorname{ran}\left(R^{-1}(T-\lambda I)R\right)^*\right]^\perp && (\ker Q = (\operatorname{ran} Q^*)^\perp \text{ for any } Q \in \mathcal{B}(\mathcal{H})) \\ &\geq \dim \mathcal{N}^\perp && \text{(since } \mathcal{N} \subset \operatorname{ran}(R^{-1}(T-\lambda I)R)^*) \\ &= \dim \overline{\operatorname{ran}(T-\lambda I)}^\perp \\ &= \dim\ker(T-\lambda I)^*. \end{aligned}$$

Therefore, $\mathrm{nul}(T-\lambda I) \geq \mathrm{nul}(T-\lambda I)^*$.

Combining the two inequalities proven above we get that $\mathrm{nul}(T-\lambda I) = \mathrm{nul}(T-\lambda I)^*$ for all $\lambda \in \mathbb{C}$. In particular, if $\lambda \in \rho_{sF}(T)$, then

$$\mathrm{ind}(T-\lambda I) = \mathrm{nul}(T-\lambda I) - \mathrm{nul}(T-\lambda I)^* = 0.$$

Therefore, T is biquasitriangular. $\square$

Now consider the result from Theorem 3.25, namely the fact that for a $\mathcal{C}^*$-algebra with the total reduction property every representation is similar to a *-representation. Suppose $\mathfrak{A} \subset \mathcal{T}_\infty$ has the total reduction property and is similar to a $\mathcal{C}^*$-algebra $\mathfrak{C}$ (we shall prove later that these two conditions are equivalent). So there is some similarity matrix S such that $\mathfrak{C} = S\mathfrak{A}S^{-1}$. Then $\mathfrak{C}$ also has the total reduction property, so by Theorem 3.25 every representation is similar to a *-representation. Suppose also that $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$ is isomorphic to $\mathfrak{A}$, with some isomorphism $\rho : \mathfrak{A} \to \mathfrak{B}$. Then $\phi : T \mapsto \rho(S^{-1}TS)$ is a representation of $\mathfrak{C}$ with range $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$. Since ϕ is similar to a *-representation we get that $\mathfrak{B}$ is similar to a $\mathcal{C}^*$-algebra. Therefore, any operator algebra isomorphic to $\mathfrak{A}$ is also similar to a $\mathcal{C}^*$-algebra.

Remark 4.17. Suppose $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ has the complete reduction property. If $\mathcal{M}$ is an invariant subspace of $\mathfrak{A}$, we can find a similarity S such that both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for $S^{-1}\mathfrak{A}S$. The construction described below will be used in multiple theorems in this section; the form of the similarity matrix S and of $S^{-1}\mathfrak{A}S$ plays an important role.

Write the operators of $\mathfrak{A}$ with respect to $\mathcal{M}$ and $\mathcal{M}^\perp$. Since $\mathcal{M}$ is invariant, we know all the operators will have the form $\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$. Let $P = \begin{bmatrix} I & P_2 \\ 0 & 0 \end{bmatrix} \in \mathfrak{A}'$ be a projection onto $\mathcal{M}$ (such a projection exists since $\mathfrak{A}$ is a complete reduction algebra, so $\mathcal{M}$ has a complementary module). Since P is in the commutant of $\mathfrak{A}$, by multiplying the matrices we get that $A_2 + P_2A_4 = A_1P_2$ for each matrix $A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} \in \mathfrak{A}$.

Define $S = \begin{bmatrix} I & -P_2 \\ 0 & I \end{bmatrix}$. Note that $S^{-1} = \begin{bmatrix} I & P_2 \\ 0 & I \end{bmatrix}$. Then we can multiply the matrices to get that if $A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ then $S^{-1}AS = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$ (note: to get this result, use the fact that $A_2 + P_2A_4 = A_1P_2$, as shown above). Hence $\mathcal{M}$ and $\mathcal{M}^\perp$ are both invariant for $S^{-1}\mathfrak{A}S$. $\square$

From the above remark it is clear that if $\mathcal{M}$ is an invariant subspace and we write $\mathfrak{A}$ with respect to $\mathcal{M}$ and $\mathcal{M}^\perp$, then $\mathfrak{A}$ cannot contain a matrix $\begin{bmatrix} 0 & T_2 \\ 0 & 0 \end{bmatrix}$ where $T_2 \neq 0$. To see this, if P is a projection onto $\mathcal{M}$ as in the proof, then we must have $T_2 + P_2 0 = 0P_2$, which gives us $T_2 = 0$.

Observe that if $\mathfrak{A}$ is an abelian, total reduction algebra, then so is $S^{-1}\mathfrak{A}S$ where S is a similarity matrix (recall that the total reduction property is preserved by homomorphisms).

Theorem 4.18. *Suppose $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is an abelian unital subalgebra with the total reduction property. If $T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, then $\ker(T - \lambda I) = \ker(T - \lambda I)^m$ for all $m \geq 2$.*

Proof. Consider $m = 2$. Let $\mathcal{M} = \ker(T - \lambda I)$ and $\mathcal{N} = \ker(T - \lambda I)^2$. We have already seen that $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}\mathfrak{A}$. Also clearly $\mathcal{M} \subset \mathcal{N}$. We want to show that $\mathcal{M} = \mathcal{N}$.

Suppose first that $\mathcal{N} = \mathcal{H}$, and assume that $\mathcal{M} \neq \mathcal{N}$. Write the matrices of $\mathfrak{A}$ with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$; we want to figure out the matrix for T. Suppose $T - \lambda I = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$. We know $A_3 = 0$ since $\mathcal{M}$ is invariant for $\mathfrak{A}$. For every $u \in \mathcal{M}$ we have $(T - \lambda I)u = 0$ (by definition of $\mathcal{M}$), and so we must have $A_1 = 0$. Consider any $v \in \mathcal{H} = \mathcal{N}$; by definition of $\mathcal{N}$, $(T - \lambda I)^2 v = 0$. Hence $(T - \lambda I)v \in \ker(T - \lambda I) = \mathcal{M}$; so the range of $T - \lambda I$ is contained in $\mathcal{M}$. From this observation we get that $A_4 = 0$ as well. So $T - \lambda I = \begin{bmatrix} 0 & T_2 \\ 0 & 0 \end{bmatrix}$. Since $\mathfrak{A}$ has the total reduction property, we also have $T_2 = 0$ (follows from the comment made after Remark 4.17). So for any $v \in \mathcal{M}^\perp$ we get $(T - \lambda I)v = 0$, i.e. $v \in \mathcal{M}$. This contradiction allows us to conclude that we must have $\mathcal{M} = \mathcal{N}$.

If $\mathcal{N} \neq \mathcal{H}$, then the algebra $\mathfrak{B} = P_{\mathcal{N}}\mathfrak{A}P_{\mathcal{N}}$ (where $P_{\mathcal{N}}$ is the orthogonal projection onto $\mathcal{N}$) has the total reduction property (since this property is preserved by homomorphisms). Moreover, we can consider $\mathfrak{B}$ as a subset of $\mathcal{B}(\mathcal{N})$; since $\mathcal{M}$ is an invariant subspace for $\mathfrak{B}$, the discussion above (for $\mathcal{N} = \mathcal{H}$) applies, and gives us $\mathcal{M} = \mathcal{N}$.

Suppose $\ker(T - \lambda I)^i = \ker(T - \lambda I)$ for some $i \geq 2$. Then $u \in \ker(T - \lambda I)^{i+1}$

means that $(T-\lambda I)u \in \ker (T-\lambda I)^i = \ker (T-\lambda I)$. So $(T-\lambda I)^2 u = 0$, and hence $u \in \ker (T-\lambda I)^2 = \ker (T-\lambda I)$. Therefore, $\ker (T-\lambda I)^{i+1} \subset \ker (T-\lambda I)$, and since the opposite inclusion is obvious equality follows. Therefore, $\ker (T-\lambda I)^m = \ker (T-\lambda I)$ for all $m \geq 2$. $\square$

Lemma 4.19. *Let $\mathfrak{A}$ be a unital, abelian, total reduction subalgebra of $\mathcal{B}(\mathcal{H})$. Consider $T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ an eigenvalue of T. Then there exists an unique projection E onto* $\ker (T-\lambda I)$. *Moreover, E is a central projection in $\mathfrak{A}''$.*

Proof. We can assume without loss of generality that $\lambda = 0$, since otherwise we can replace T by $(T-\lambda I) \in \mathfrak{A}$. Let $\mathcal{M} = \ker T$ and E be a projection onto $\mathcal{M}$. Recall that $\mathcal{M} \in \operatorname{Lat}\mathfrak{A}$, so $T = \begin{array}{c} \begin{bmatrix} \lambda I & T_2 \\ 0 & T_4 \end{bmatrix} \end{array} \begin{array}{c} \mathcal{M} \\ \mathcal{M}^\perp \end{array} = \begin{bmatrix} 0 & T_2 \\ 0 & T_4 \end{bmatrix}$.

We want to show that $\ker T_4 = \{0\}$ and $\operatorname{ran} T_4$ is dense in $\mathcal{M}^\perp$. By the comments in Remark 4.17 we can find a similarity S such that $S^{-1}TS = \begin{bmatrix} 0 & 0 \\ 0 & T_4 \end{bmatrix}$ and $S^{-1}ES = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Now if $u \in \ker T_4$, then $\begin{bmatrix} 0 \\ u \end{bmatrix} \in \ker S^{-1}TS$. It follows that $S^{-1}\begin{bmatrix} 0 \\ u \end{bmatrix} \in \ker T = \mathcal{M}$. Recall however from Remark 4.17 that $S^{-1} = \begin{bmatrix} I & P_2 \\ 0 & I \end{bmatrix}$ for some module map $P_2 : \mathcal{M}^\perp \to \mathcal{M}$. Multiplying this out we get that $\begin{bmatrix} P_2 u \\ u \end{bmatrix} \in \mathcal{M}$, and hence $u = 0$. Therefore, $\ker T_4 = \{0\}$.

Now by contradiction suppose that $\operatorname{ran} T_4$ is not dense in $\mathcal{M}^\perp$, and let $\mathcal{N} = \overline{\operatorname{ran} T_4}$. Then with respect to the decomposition $\mathcal{M} \oplus \mathcal{N} \oplus (\mathcal{M}^\perp \ominus \mathcal{N})$ the matrix for $S^{-1}TS$ looks like $\begin{bmatrix} 0 & 0 & 0 \\ 0 & T_{4,1} & T_{4,2} \\ 0 & 0 & 0 \end{bmatrix}$. Note that $\overline{\operatorname{ran} S^{-1}TS} = \mathcal{M} \oplus \mathcal{N}$. Hence $\mathcal{M} \oplus \mathcal{N}$ is an invariant subspace for $S^{-1}\mathfrak{A}S$, and since $S^{-1}\mathfrak{A}S$ has the total reduction property (the property is preserved by similarities), we can apply the construction in Remark 4.17 again to find a similarity U such that $U^{-1}(S^{-1}TS)U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & T_{4,1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with $U^{-1} = \begin{bmatrix} I & Q_2 \\ 0 & I \end{bmatrix} \begin{array}{c} (\mathcal{M} \oplus \mathcal{N}) \\ (\mathcal{M} \oplus \mathcal{N})^\perp \end{array}$ for some module map Q_2. Clearly any $w \in (\mathcal{M}^\perp \ominus \mathcal{N})$ is in the kernel of $U^{-1}S^{-1}TSU$, and hence $U^{-1}S^{-1}w \in \mathcal{M}$. With respect to the decomposition $\mathcal{M} \oplus \mathcal{N} \oplus (\mathcal{M}^\perp \ominus \mathcal{N})$ we have $U^{-1} = \begin{bmatrix} I & 0 & Q_{1,2} \\ 0 & I & Q_{2,2} \\ 0 & 0 & I \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} I & P_{2,1} & P_{2,2} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, so

$$U^{-1}S^{-1}\begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} P_{2,2}w + T_{2,1}w \\ T_{2,2}w \\ w \end{bmatrix}$$

This vector is in $\mathcal{M}$ only if $w = 0$, so $\mathcal{M}^\perp \ominus \mathcal{N} = \{0\}$. Therefore, $\mathcal{N} = \mathcal{M}^\perp$.

Consider any $R \in (S^{-1}\mathfrak{A}S)'$, say $R = \begin{array}{c} \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \begin{array}{c} \mathcal{M} \\ \mathcal{M}^\perp \end{array} \end{array}$. Since R commutes with $S^{-1}TS$, the matrix multiplication $R(S^{-1}TS) = (S^{-1}TS)R$ gives us that $R_2T_4 = 0$, and $0 = T_4R_3$. Using the fact that T_4 is injective and has dense range we can conclude that $R_2 = R_3 = 0$. So $R = \begin{bmatrix} R_1 & 0 \\ 0 & R_4 \end{bmatrix}$. Recall that $S^{-1}ES = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; this matrix clearly commutes with R. Hence $S^{-1}ES \in (S^{-1}\mathfrak{A}S)''$. Therefore, $E \in \mathfrak{A}''$.

Finally, suppose $P \in \mathfrak{A}'$ is a projection onto $\mathcal{M}$. Then P commutes with E (since $E \in \mathfrak{A}''$) and the range of P is the same as that of E, so $P = E$. Therefore, $E \in \mathfrak{A}' \cap \mathfrak{A}''$ is the unique projection onto $\mathcal{M}$. □

Lemma 4.20. *Let $\mathfrak{A}$ be an abelian, total reduction subalgebra of $\mathcal{B}(\mathcal{H})$. Consider $T \in \mathfrak{A}$. Let Λ be the set of eigenvalues of T. For $\lambda \in \Lambda$, denote by E_λ the projection onto* $\ker T - \lambda I$ *described in Lemma 4.19. Then $\mathcal{F} = \{\sum_{i=1}^{n} E_{\lambda_i} : n \in \mathbb{N}, \lambda_i \in \Lambda\}$ is a bounded set of commuting idempotents closed under symmetric differences. Moreover, if $\mathcal{H}$ is separable, it follows that Λ is countable.*

Proof. Consider eigenvalues λ and μ such that $\lambda \neq \mu$. Recall from Lemma 4.19 that $E_\lambda, E_\mu \in \mathfrak{A}' \cap \mathfrak{A}''$. It follows that E_λ and E_μ commute. Then, as a consequence of Lemma 3.2, there exists a similarity S which orthogonalizes E_λ and E_μ. Let $F_\lambda = SE_\lambda S^{-1}$ and $F_\mu = SE_\mu S^{-1}$. The range of F_λ is $S\ker(T - \lambda I)S^{-1}$ and the range of F_μ is $S\ker(T - \mu I)S^{-1}$; the two sets are clearly disjoint. Thus F_λ and F_μ are commuting orthogonal projections with disjoint ranges, so we have $F_\lambda F_\mu = 0$. But $(SE_\lambda S^{-1})(SE_\mu S^{-1}) = 0$ implies $E_\lambda E_\mu = 0$. Therefore, we have shown that, whenever $\lambda \neq \mu$ are two eigenvalues in Λ, we get $E_\lambda E_\mu = 0$. It follows immediately that any element of $\mathcal{F}$ is an idempotent.

Suppose $F_\lambda = \sum_{i=1}^{n} E_{\lambda_i}$ and $F_\mu = \sum_{j=1}^{m} E_{\mu_j}$ are two operators in $\mathcal{F}$. Since the E_λ's and the E_μ's commute, it follows that F_λ and F_μ commute as well. So all the idempotents in $\mathcal{F}$ commute. Moreover, $F_\lambda + F_\mu - 2F_\lambda F_\mu = \sum_{i=1}^{n} E_{\lambda_i} + \sum_{j=1}^{m} E_{\mu_j} - 2 \sum_{\substack{i=1\ldots n \\ j=1\ldots m}} E_{\lambda_i} E_{\mu_j}$. We can use the fact that $E_{\lambda_i} E_{\mu_j} = 0$ if $\lambda_i \neq \mu_j$ and to E_{λ_i} if $\lambda_i = \mu_j$ to simplify the sum to $\sum_{\substack{i=1\ldots n \\ \lambda_i \notin \{\mu_j\}_{j=1}^{m}}} E_{\lambda_i} + \sum_{\substack{j=1\ldots m \\ \mu_j \notin \{\lambda_i\}_{i=1}^{n}}} E_{\mu_j}$, which is clearly an element of $\mathcal{F}$. Therefore, $\mathcal{F}$ is closed under symmetric differences.

Finally, we want to show that $\mathcal{F}$ is a bounded set. Consider any element $F_\lambda = \sum_{i=1}^{n} E_{\lambda_i}$ in $\mathcal{F}$. Since E_{λ_i} belongs to $\mathfrak{A}' \cap \mathfrak{A}''$ for each i, so does F_λ. Also, ran $F_\lambda = \mathit{span}\{\text{ran} E_{\lambda_i}\}_{i=1}^{n}$ (note that ran F_λ is closed since ran E_{λ_i} is closed for each i and as explained earlier there is a similarity matrix which makes the E_{λ_i}'s orthogonal). But then, similar to the earlier proof that E_{λ_i} is unique (see Theorem 4.19), F_λ is the unique projection in $\mathfrak{A}'$ onto ran F_λ. Since $\mathfrak{A}$ has the total reduction property, it follows that $\{\|E\|\}_{E \in \mathcal{F}}$ is bounded by the projection constant of $\mathfrak{A}$ (see Theorem 3.19).

If $\mathcal{H}$ is separable, then there are only countably many mutually orthogonal projections in $\mathcal{B}(\mathcal{H})$. But we have shown earlier that if λ and μ are two distinct eigenvalues in Λ, then there is a similarity matrix S for which $SE_\lambda S^{-1}$ and $SE_\mu S^{-1}$ are mutually orthogonal projections. In fact, since $\mathcal{F}$ is a bounded set of commuting idempotents closed under symmetric differences, Lemma 3.2 gives us that the same S can be used for all the eigenvalues in Λ. Hence the set $\{SE_\lambda S^{-1} : \lambda \in \Lambda\}$ is a subset of the set of mutually orthogonal projections of $\mathcal{B}(\mathcal{H})$, and as such is countable. Therefore, Λ is countable.

By Lemma 3.2 we can find a similarity matrix S such that SPS^{-1} is self-adjoint for each $P \in \mathcal{F}$. □

Now suppose that $\mathfrak{A} \subset \mathcal{T}_\infty$ is a total reduction algebra. Denote by $\{r_{ii}\}$ the diagonal entries of R. We will show that each r_{ii} is an eigenvalue of R. Since R is upper triangular, e_1 is an eigenvector of R corresponding to r_{11}. For $i > 1$, consider the subspace $\mathcal{H}_{i-1} = \mathit{span}\{e_1, \ldots, e_{i-1}\}$. This is an invariant subspace for $\mathfrak{A}$; hence, by Remark 4.17, we can find a similarity S for which $SRS^{-1} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & r_{ii} & R_2 \\ 0 & 0 & R_3 \end{bmatrix}$ (where the R_i's are corresponding submatrices from the original matrix R). It follows immediately that r_{ii} is an eigenvalue of R which has $S^{-1}e_i$ as an eigenvector. Denote by $\lambda_1, \lambda_2, \ldots$ the distinct values in $\{r_{ii}\}$. Using Theorem 4.18, we can show that $\overline{span}\{\ker(R - \lambda_i I) : i \in \mathbb{N}\} = \mathcal{H}$, so the eigenvectors of R span $\mathcal{H}$. Then we can find a decomposition of $\mathcal{H}$ with respect to which we get

$$R = \begin{array}{c} \begin{pmatrix} \lambda_1 I_1 & R_{12} & R_{13} & \ldots \\ 0 & \lambda_2 I_2 & R_{23} & \ldots \\ 0 & 0 & \lambda_3 I_3 & \ldots \\ 0 & 0 & 0 & \ddots \end{pmatrix} \end{array} \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \end{array}$$

In order to prove our main result for $\mathfrak{A} \subset \mathcal{T}_\infty$ we will also need the following theorem.

Theorem 4.21. *[[8], Theorem 4.2.1] Suppose $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is an abelian, total reduction algebra and $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$ is an abelian $\mathcal{C}^*$-algebra such that $\mathfrak{A} \subset \mathfrak{B}$. Then $\mathfrak{A}$ is self-adjoint.*

Finally, we are able to prove the main theorem of this section.

Theorem 4.22. *Suppose $\mathfrak{A} \subseteq \mathcal{T}_\infty$ is a unital Banach algebra. Then the following are equivalent:*
a) $\mathfrak{A}$ is a total reduction algebra.
b) $\mathfrak{A}$ is amenable.
c) $\mathfrak{A}$ is similar to an abelian $\mathcal{C}^*$-algebra.

Proof. $c) \Rightarrow b)$ Recall that all abelian $\mathcal{C}^*$-algebras are amenable. In particular, if we let $\mathfrak{B}$ be the abelian $\mathcal{C}^*$-algebra similar to $\mathfrak{A}$, then $\mathfrak{B}$ is amenable. Since the similarity matrix allows us to define a continuous homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ whose range is $\mathfrak{A}$, by Theorem 2.19 it follows that $\mathfrak{A}$ must be amenable.

$b) \Rightarrow a)$ All amenable algebras are total reduction algebras, as shown in the comment following Theorem 3.15.

$a) \Rightarrow c)$ Suppose $\mathfrak{A}$ is a total reduction algebra. Then by Theorem 4.15, $\mathfrak{A}$ is abelian. Given $T \in \mathfrak{A}$ denote by π_T the identity representation of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$, with the understanding that $\pi_T(T)$ is assumed to have the form described in the comment following Theorem 4.20. Then each π_T is injective and $\{\|\pi_T\|\}$ is bounded.

Consider the Gelfand map $\Gamma : \mathfrak{A} \to \mathcal{C}(\Sigma_{\mathfrak{A}})$. We will show that Γ is injective, has dense range and is bounded below; hence Γ is invertible, and so it is an isomorphism from $\mathfrak{A}$ to $\mathcal{C}(\Sigma_{\mathfrak{A}})$. This will allow us to conclude that $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra.

To show that Γ is injective, we will show that $\ker \Gamma = \{0\}$. Recall that for $A \in \mathfrak{A}$ $\operatorname{ran}\Gamma(A) = \operatorname{spr}(A)$, so the kernel of Γ is the set of quasinilpotent operators of $\mathfrak{A}$. Consider $Q \in \mathfrak{A}$ a quasinilpotent operator. Then the only possible eigenvalue of Q is 0, so by looking at the form of $\pi_Q(Q)$ it must consist of a single block λI where $\lambda = 0$. Thus $\pi_Q(Q) = 0$. But π_Q is injective, so we must have $Q = 0$. Therefore, $\ker \Gamma = \{0\}$.

The total reduction algebra is preserved by homomorphisms (Theorem 3.18), so $\overline{\Gamma(\mathfrak{A})} \subset \mathcal{C}(\Sigma_{\mathfrak{A}})$ is an abelian, total reduction algebra. Hence by Theorem 4.21, $\overline{\Gamma(\mathfrak{A})}$ is self-adjoint. It also contains the constants and separates the points of $\Sigma_{\mathfrak{A}}$ (properties of the Gelfand map), so the Stone-Weierstrass Theorem gives us that $\overline{\Gamma(\mathfrak{A})} = \mathcal{C}(\Sigma_{\mathfrak{A}})$.

Now we shall show that Γ is bounded below. Since $\{\pi_T : T \in \mathfrak{A}\}$ is bounded, by Theorem 3.24 we know that we can find a constant K such that for every submodule of $\pi_T(\mathfrak{A})$ there is a projection onto the submodule with norm at most K. Fix $T \in \mathfrak{A}$ and denote by λ_i the distinct diagonal entries in $\pi_T(T)$. From the comment following Theorem 4.20, we know that the λ_i's are eigenvalues of T whose eigenvectors span $\mathcal{H}$. Let $\mathcal{M}_i = \ker \pi_T(T) - \lambda_i I$. Let $\mathcal{F}$ be the set of projections E_i onto $\mathcal{M}_i$, as described in Theorem 4.19. We know that E_i is the unique projection onto $\mathcal{M}_i$. So since a projection onto $\mathcal{M}_i$ with norm at most K must exist, we have that $\|E_i\| \leq K$. Recall from Theorem 4.19 that $\mathcal{F}$ is a set of commuting projections closed under symmetric differences. From above, $\mathcal{F}$ is bounded by K. So by Theorem 3.2 there is a similarity S such that $\|S^{-1}\|\|S\| \leq (1+2K)^2$ and $S^{-1}E_iS$ is self-adjoint. Also $S^{-1}E_iS$ commutes with $S^{-1}TS$ (since $E_i \in \mathfrak{A}'$). It follows that $S^{-1}TS = diag\,\{t_i I_{\mathcal{H}_i}\}$. So $\operatorname{spr}(S^{-1}TS) = \sup|t_i| \leq \operatorname{spr}(T)$. But then

$$\begin{aligned}\|T\| = \|SS^{-1}TSS^{-1}\| &\leq \|S\|\|S^{-1}\|\|S^{-1}TS\| \\ &\leq (1+2K)^2\|diag\,\{t_i I_{\mathcal{H}_i}\}\| \\ &\leq (1+2K)^2\operatorname{spr}(T) \\ &\qquad \text{(since } t_i \text{ is an eigenvalue of } T\text{)}\end{aligned}$$

Recall that K does not depend on T, and that $\|\Gamma(T)\| = \operatorname{spr}(T)$; so Γ is bounded below.

Therefore, Γ is invertible. Then Γ^{-1} is a representation from the abelian total reduction $\mathcal{C}^*$-algebra $\mathcal{C}(\Sigma_{\mathfrak{A}})$ to $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$, so by Theorem 3.25 it is similar to a *-representation.

It follows that $\mathfrak{A}$ is similar to a $\mathcal{C}^*$-algebra, as desired. □

4.3 Concluding Remarks

We have shown that total reductivity is a necessary and sufficient condition for $\mathfrak{A}$ to be similar to a $\mathcal{C}^*$-algebra in the cases where $\mathfrak{A}$ is an algebra of compact operators or an abelian algebra of triangular operators. In each of these cases $\mathfrak{A}$ also proved to be amenable.

However, recall that the class of totally reductive algebras is strictly larger than that of amenable algebras. Hence further research is needed, in particular to check if the condition that if an algebra $\mathfrak{A}$ is abelian is sufficient for the algebra to be totally amenable (though such a condition is clearly not necessary), and in general, to find out what conditions one can place on $\mathfrak{A}$ such that amenability implies total reductivity.

It is to be hoped that a better understanding of the properties of amenability and total reductivity will eventually lead us to a complete description of the operator algebras which are similar to $\mathcal{C}^*$-algebras.

Bibliography

[1] C. Apostol, C. Foiaş, and D. Voiculescu. Some results on non-quasitriangular operators iv. *Rev. Roum. Math. Pures et Appl.*, 18:487 – 514, 1973.

[2] D. P. Blecher and C. Le Merdy. *Operator Algebras and Their Modules.* London Mathematical Society Monographs (New Series). Oxford University Press, Oxford, 2004.

[3] H. F. Bohnenblust and S. Karlin. Geometrical properties of the unit sphere of banach algebras. *The Annals of Mathematics, 2nd Ser.*, 62(2):217–229, 1955.

[4] A. Connes. On the cohomology of operator algebras. *J. Func. Anal.*, 28:248 – 253, 1978.

[5] J. B. Conway. *A Course in Operator Theory*, volume 21 of *Graduate Studies in Mathematics*. American Mathematics Society, Providence, 2000.

[6] K. R. Davidson. *Nest algebras.* Longman Group UK Limited, Essex, 1988.

[7] G. B. Folland. *A Course in Abstract Harmonic Analysis.* CRC Press, Boca Raton, FL, 1995.

[8] J. A. Gifford. *Operator algebras with a reduction property.* Phd dissertation, Australian National University, 1997.

[9] U. Haagerup. All nuclear C^* algebras are amenable. *Invent. Math.*, 74:305 – 319, 1983.

[10] P. R. Halmos. *A Hilbert Space Problem Book.* Springer-Verlag, New York, 1982.

[11] D. A. Herrero. *Approximation of Hilbert Space Operators, Volume 1, 2nd edition*, volume 224 of *Pitman Reserach Notes in Mathematics*. Jon Wiley and Sons, New York, 1989.

[12] R. V. Kadison. On the orthogonalization of operator representations. *American Journal of Mathematics*, 77:600 – 620, 1955.

[13] L. W. Marcoux. On abelian, triangularisable, total reduction algebras. Preprint.

[14] A. L. Paterson. *Amenability*, volume 29 of *Math. Surveys and Monographs*. American Math. Soc., Providence, RI, 1988.

[15] G. Pisier. *Similarity Problems and Completely Bounded Maps*. Springer-Verlag, New York, 1996.

[16] G. Pisier. Are unitarizable groups amenable? In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progress in Mathematics*, pages 323 – 362. Birkhäuser, Basel, 2005.

[17] H. Radjavi and P. Rosenthal. *Simultaneous Triangularization*. Springer-Verlag, New York, 2000.

[18] H. Radjavi and P. Rosenthal. *Invariant Subspaces*. Dover Publications, Inc., Mineola, second edition, 2003.

[19] R. Raghavendran. Toeplitz-hausdorff theorem on numerical ranges. *Proceedings of the American Mathematical Society*, 20(1):284–285, 1969.

[20] V. Runde. *Lectures in Amenability*. Springer-Verlag, New York, 2002.

[21] I. Schochetman, R. Smith, and S. Tsui. On the closure of the sum of closed subspaces. *Int. J. Math. Math. Sci.*, 26(5):257–267, 2001.

[22] D. van Dulst. *Characterizations of Banach spaces not containing l^1*. Stichting Mathematisch Centrum, Amsterdam, 1989.